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Published by IOP Publishing for SISSA

Received: *January 15, 2009*
Accepted: *January 30, 2009*
Published: *February 17, 2009*

Covariant field equations, gauge fields and conservation laws from Yang-Mills matrix models

Harold Steinacker

Fakultät für Physik, Universität Wien,
Boltzmanngasse 5, A-1090 Wien, Austria
E-mail: harold.steinacker@univie.ac.at

Abstract: The effective geometry and the gravitational coupling of nonabelian gauge and scalar fields on generic NC branes in Yang-Mills matrix models is determined. Covariant field equations are derived from the basic matrix equations of motions, known as Yang-Mills algebra. Remarkably, the equations of motion for the Poisson structure and for the nonabelian gauge fields follow from a matrix Noether theorem, and are therefore protected from quantum corrections. This provides a transparent derivation and generalization of the effective action governing the SU(n) gauge fields obtained in [1], including the would-be topological term. In particular, the IKKT matrix model is capable of describing 4-dimensional NC space-times with a general effective metric. Metric deformations of flat Moyal-Weyl space are briefly discussed.

Keywords: Brane Dynamics in Gauge Theories, Non-Commutative Geometry, M(atrix) Theories, Gauge Symmetry.

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1. Introduction

This paper is a continuation of a series of papers [1, 3, 4, 2] with the aim to understand the geometry and the physics on noncommutative (NC) branes in Matrix-Models of Yang-Mills type. The motivation of this enterprise is to study whether such branes can serve as dynamical space-time including matter, gauge fields and gravity. In the previous papers, it was shown that these matrix models provide indeed a mechanism for emergent gravity on the branes, with an effective metric which couples universally (up to possibly conformal factors) to bosonic and fermionic matter as well as nonabelian gauge fields. The mechanism of gravity on these branes is different from general relativity a priori, and related to gauge theory on non-commutative spaces. This is very similar in spirit to the ideas put forward in [5 − 7]; see also e.g. [8, 10, 9] for related work.

The starting point of our approach are matrix models of Yang-Mills type, with bosonic action

$$S_{\rm YM} = -Tr[X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'} \tag{1.1}$$

where η_{ab} is a constant background metric with Euclidean or Minkowski signature. These models arise e.g. through dimensional reduction of U($\mathcal{N}$) Yang-Mills gauge theory with $\mathcal{N} \to \infty$ to zero dimensions. The models describe in particular noncommutative (NC) branes of even dimension $2n$, including e.g. the Moyal-Weyl quantum plane

$[X^\mu, X^\nu] = i\theta^{\mu\nu}\mathbb{1}$ which is a solution of the equations of motion. It was generally believed that fluctuations of the matrices or "covariant coordinates" X^a describe gauge fields on such NC spaces. However, a detailed analysis [1, 5] shows that the U(1) components of these fluctuations are actually gravitational degrees of freedom, and only the SU(n) components should be interpreted as nonabelian gauge fields. This provides a detailed explanation of UV/IR mixing in NC gauge theory in terms of an induced gravitational action [3]. The effective action for the SU(n) gauge fields was found in [1] using a brute-force computation. It was shown that the gauge fields couple as expected to the effective gravitational metric, and an interesting "would-be topological" term in the action was found.

The effective metric on the NC branes was generalized in [2] to the case of non-trivially embedded branes $\mathcal{M}_\theta \subset \mathbb{R}^D$ in higher-dimensional matrix models, and has the form $\tilde{G}^{\mu\nu} = e^{-\sigma}\theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)g_{\mu'\nu'}(x)$. This is very similar to the open string metric in the context of string theory [11], while the "embedding metric" $g_{\mu'\nu'}(x)$ corresponds to the closed string metric and enters the action on the branes only indirectly. Indeed it appears that only $D > 4$ can provide a large enough class of effective metrics to allow a realistic description of gravity. The case $D = 10$ is favored from the point of view of quantization, more precisely the IKKT model [12] which was introduced originally in the context of string theory; this allows to describe the most general effective metric on 4-dimensional branes. However, the coupling of nonabelian gauge fields to this general metric was not established up to now.

In the present paper, we study in detail gauge fields on generic NC branes $\mathcal{M}_\theta \subset \mathbb{R}^D$ with $D > 4$ governed by the matrix model (1.1). SU(n) gauge fields are shown to arise on $\mathcal{M}_\theta$ as non-abelian fluctuations of the matrices or "covariant coordinates" X^a, and couple as expected to the effective metric $\tilde{G}^{\mu\nu}$. We provide a new derivation of the Yang-Mills equations of motion (5.16) for nonabelian gauge fields coupled to $\tilde{G}^{\mu\nu}$ on general NC branes, as well as the equation governing the noncommutativity resp. Poisson tensor $\theta^{\mu\nu}(x)$. Remarkably, both equations can be understood as consequences of a basic "translational" symmetry of the underlying matrix model $X^a \to X^a + c^a\mathbb{1}$. The corresponding Noether theorem yields precisely the above equations. This provides not only a transparent derivation of these equations of motion; as usual in quantum field theory, a derivation based on a fundamental symmetry also implies that these equations are protected from quantum corrections. Therefore these equations of motion are valid also at the quantum level. In particular, (2.40) provides the relation between non-commutativity of space-time and gravity.

Form a mathematical point of view, it is quite remarkable that the covariant Yang-Mills equations (5.16) as well as (2.40) are direct consequences of the underlying matrix equations of motion known as Yang-Mills algebra [14, 13],

$$[X^a, [X^b, X^{a'}]]g_{aa'} = 0 \tag{1.2}$$

in the background of a quantized Poisson manifold. The resulting structure is very rigid and leaves little room for different interpretations. Combined with the well-known fact [15] that any analytic 4-dimensional (pseudo) Riemannian manifold $(\mathcal{M}^4, g_{\mu\nu})$ can be embedded locally isometrically in flat (pseudo)Euclidean space $\mathbb{R}^{10}$, it follows that the Riemannian geometry required for gravity can be described in terms of a simple matrix model of the

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above type, notably the IKKT model with $D = 10$. These models therefore provide a possible foundation of geometry and gravity, without any classical-geometrical prerequisites: the geometry arises only effectively in a semi-classical limit. Moreover, gravitons are naturally unified with (nonabelian) gauge fields, both being described by fluctuations of the underlying matrices around some given background. This provides a simple and promising framework for a unified description for gauge fields, gravity and matter. Moreover, it is reasonable to expect that the IKKT model is well-defined at quantum level, due to its extended supersymmetry. This supports the hypothesis that these matrix models may be a suitable framework to describe the quantum regime of gravity.

Some new results on various aspects of the geometry of the NC branes are also given. In particular, we elaborate in some detail the preferred frame and gauge defined by the matrix model. This provides a simplified derivation of the fact that the linearized metric fluctuations around flat Moyal-Weyl space due to the would-be U(1) gauge fields are Ricci-flat but non-trivial, as first observed by Rivelles [5]; they are hence interpreted as gravitons. We give explicit expressions for the linearized Riemann tensors, including also fluctuations of the embedding. Nevertheless, the physical aspects of the emergent gravity theory are not sufficiently well understood at this point and require much more work.

The results of this paper illustrate the unexpected role of symmetries in emergent gravity. The classical concepts of diffeomorphism resp. general coordinate invariance loose their significance, being partially replaced by symplectomorphism invariance; see also [7] for related discussion from a somewhat different point of view. On the other hand, novel symmetries such as the above "translational" symmetry arise with unexpected consequences. In particular, we point out that the global $\mathrm{SO}(D)$ symmetry of the model can be exploited by going to certain "normal embedding coordinates" where some of the analysis simplifies.

There is of course a relation with previous work in the context of matrix models in string theory (e.g. [16, 17, 12, 18, 21, 22, 20, 23, 25, 24, 19]) and NC gauge theory (e.g. [26, 28, 27]). In particular, evidence for gravity on NC branes was obtained e.g. in [29 – 31]. However, most of this work is focused on BPS or highly symmetric (typically "fuzzy") brane solutions. The essential point of the present approach is to study generic curved NC branes and their effective geometry. The proper separation of gravitational and gauge degrees of freedom on these branes in the matrix model was understood only recently. Moreover, it turns out that all fields which arise due to fluctuations around such a background *only* live on the brane $\mathcal{M}^{2n}_\theta \subset \mathbb{R}^D$; there really is no D-dimensional "bulk" which could carry physical degrees of freedom, not even gravitons. Therefore the present framework is quite different from the braneworld-scenarios such as [32, 33]. One could argue that the strength of string theory (notably the good behavior under quantization) are preserved while the main problems (lack of predictivity) are avoided here. Clearly all essential ingredients for physics are available, but it remains to be seen whether realistic physics can be described through these models. Finally, even though we focus on the semi-classical limit, higher-order corrections in θ could and should be computed eventually. This will provide a link with NC field theory [34 – 36] and possibly with other approaches to NC gravity, see e.g. [39, 37, 38].

This paper is organized as follows. We start in section 2 by recalling the basic aspects

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of NC branes in matrix models under considerations. The frame which arises naturally is discussed in some detail in section 2.2. A preferred class of coordinates is discussed in section 2.3, taking advantage of the underlying SO(D) symmetry of the model. Metric fluctuations of flat Moyal-Weyl space are discussed in section 3.1 In section 4, the basic Noether theorem based on the translational symmetry is exploited for the U(1) sector. Nonabelian gauge fields are finally introduced in section 5, and the effective action is established based on Noether's theorem.

2. The matrix model

Consider first the basic matrix model

$$S_{\rm YM} = -Tr[X^\mu, X^\nu][X^{\mu'}, X^{\nu'}]\eta_{\mu\mu'}\eta_{\nu\nu'}, \tag{2.1}$$

for

$$\eta_{\mu\mu'} = \mathrm{diag}(1,1,1,1) \quad \text{or} \quad \eta_{\mu\mu'} = \mathrm{diag}(-1,1,1,1) \tag{2.2}$$

in the Euclidean[1] resp. Minkowski case. The X^μ for $\mu = 1,2,3,4$ are hermitian matrices ("covariant coordinates") or operators acting on some Hilbert space $\mathcal{H}$. The above action is invariant under the following fundamental gauge symmetry

$$X^\mu \to U^{-1} X^\mu U, \qquad U \in \gamma \mathrm{U}(\mathcal{H}). \tag{2.3}$$

We will denote the commutator of 2 matrices as

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \tag{2.4}$$

where $\theta^{\mu\nu} \in L(\mathcal{H})$ is *not* necessarily proportional to $1\!\!1_{\mathcal{H}}$. We focus here on configurations X^μ which can be interpreted as quantizations of coordinate functions x^μ on a Poisson manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$ with general Poisson structure $\theta^{\mu\nu}(x)$. The simplest case is the Moyal-Weyl quantum plane and its deformations discussed in section 3, but essentially any Poisson manifold provides (locally) a possible background X^μ [40]. Formally, this means that there is an isomorphism[2] of vector spaces

$$\begin{aligned} \mathcal{C}(\mathcal{M}) &\to \mathcal{A} \subset L(\mathcal{H}) \\ f(x) &\mapsto \hat{f}(X) \\ \text{such that} \quad i\{f,g\} &\mapsto [\hat{f},\hat{g}] + O(\theta^2) \end{aligned} \tag{2.5}$$

Here $\mathcal{C}(\mathcal{M})$ denotes some space of functions on $\mathcal{M}$, and $\mathcal{A}$ is interpreted as quantized algebra of functions on $\mathcal{M}$.

In this paper we only consider the *semi-classical* or *geometrical* limit of such a quantum space. This means that the space is described in terms of functions on $\mathcal{M}$ using (2.5), keeping only the Poisson bracket on the rhs of (2.5) and dropping all higher-order terms in

[1] The reason for this notation is to avoid using $g_{\mu\nu}$ for this fixed "metric"; $g_{\mu\nu}$ will arise later.

[2] operator-technical subtleties concerning well-behaved subalgebras etc. are not important here and will be ignored. In particular, x^μ resp. X^μ are considered as elements of $\mathcal{C}(\mathcal{M})$ resp. $\mathcal{A}$

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θ. We will accordingly replace $[\hat{f}(X), \hat{g}(X)] \to i\{f(x), g(x)\}$ and $[X^\mu, X^\nu] \to i\theta^{\mu\nu}(x)$. In particular,

$$[X^\mu, f(X)] \sim i\theta^{\mu\nu}(x)\frac{\partial}{\partial x^\nu} f(x) \tag{2.6}$$

which will be used throughout this paper, denoting with $\sim$ the leading contribution in a semi-classical expansion in powers of $\theta^{\mu\nu}$.

2.1 Noncommutative branes and extra dimensions

We now review the generalization to NC branes in higher-dimensional matrix models as discussed in [2]. The basic tool to understand the physics is to add a scalar ("test") field to the model. Using the above assumptions and preserving the gauge symmetry (2.3), a scalar fields can be coupled to the above matrix model by adding an action of the form

$$Tr[X^\mu, \phi][X^\nu, \phi]\eta_{\mu\nu} \;\sim\; -Tr\eta_{\mu\nu}\theta^{\mu\mu'}\theta^{\nu\nu'}\partial_\mu\phi\partial_\nu\phi. \tag{2.7}$$

The combined action has the same form as the matrix model (2.1) if we consider $\phi \equiv X^5$ as extra coordinate, but insisting that $\phi = \phi(X^\mu)$ is a function of the other 4 coordinates. This motivates to consider more generally

$$S_{\rm YM} = -Tr[X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'}, \tag{2.8}$$

for hermitian matrices or operators $X^a,\ a = 1, \ldots, D$ acting on some Hilbert space $\mathcal{H}$, and equations of motion

$$[X^a, [X^b, X^{a'}]]\eta_{aa'} = 0. \tag{2.9}$$

A scalar field can therefore be interpreted as defining an embedding of a 4-dimensional manifold (a "3-brane") in a higher-dimensional space. This suggests to consider a higher-dimensional version of the Yang-Mills matrix model, such as the IKKT model with $D = 10$.

We now to consider generic $2n$ -dimensional noncommutative spaces $\mathcal{M}_\theta^{2n} \subset \mathbb{R}^D$ (a $2n-$ 1 brane), interpreted as (Euclidean or Minkowski) space-time embedded in D dimensions. $2n \geq 4$ is allowed in order to cover configurations such as $\mathcal{M}_\theta^{2n} \cong \mathcal{M}_\theta^4 \times K_\theta \subset \mathbb{R}^D$; however we focus mainly on the case $2n = 4$. To realize this, consider the matrix model (2.8) and split[3] the matrices as

$$X^a = (X^\mu, \phi^i), \qquad \mu = 1, \ldots, 2n, \;\; i = 1, \ldots, D - 2n \tag{2.10}$$

where the "scalar fields" $\phi^i = \phi^i(X^\mu)$ are assumed to be functions of X^μ. The basic example is a flat embedding of a 4-dimensional NC background

$$\begin{aligned} [X^\mu, X^\nu] &= i\theta^{\mu\nu}, \qquad && \mu, \nu = 1, \ldots, 4, \\ \phi^i &= 0, && i = 1, \ldots, D - 4 \end{aligned} \tag{2.11}$$

[3]For generic embeddings, the separation (2.10) is arbitrary, and we are free to choose different $2n$ components among the $\{X^a\}$ as generators of tangential vector fields. This corresponds to a very interesting transformation exchanging fields with coordinates, related to the discussion in section 2.3.

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where X^μ generates a 4-dimensional NC space $\mathcal{M}^4_\theta$. In the semi-classical limit, suitable "optimally localized states" of $\langle X^a\rangle \sim x^a$ will then be located on $\mathcal{M}^{2n} \subset \mathbb{R}^D$. One could interpret $\phi^i(x)$ as scalar fields on $\mathbb{R}^4_\theta$ generated by X^μ; however, it is more appropriate to interpret $\phi^i(x)$ as purely geometrical degrees of freedom, defining the embedding of a $2n$ - dimensional submanifold $\mathcal{M}^{2n} \subset \mathbb{R}^D$. This $\mathcal{M}^{2n}$ carries the induced metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^j\delta_{ij} \;=\; \partial_\mu x^a\partial_\nu x^b\,\eta_{ab} \tag{2.12}$$

via pull-back of η_{ab}. Note that $g_{\mu\nu}(x)$ is *not* the metric responsible for gravity, and will enter the action only implicitly. We will see below that all fields coupling to such a background will *only* live on the brane $\mathcal{M}^{2n}$, and there really is no higher-dimensional "bulk" which could carry physical degrees of freedom, not even gravitons. Therefore the present framework is quite different from the braneworld-scenarios such as [32, 33].

Poisson and metric structures. Expressing the ϕ^i in terms of X^μ, we obtain

$$[\phi^i, f(X^\mu)] \;\sim\; i\theta^{\mu\nu}\partial_\mu\phi^i\partial_\nu f = ie^\mu(f)\,\partial_\mu\phi^i \tag{2.13}$$

in the semi-classical limit. This involves only the components $\mu = 1,\ldots,2n$ of the anti-symmetric "tensor" $[X^a, X^b] \sim i\theta^{ab}(x)$, which has rank $2n$ in the semi-classical limit. Here

$$e^\mu := -i[X^\mu, .] \;\sim\; \theta^{\mu\nu}\partial_\nu \tag{2.14}$$

are derivations, which span the tangent space of $\mathcal{M}^{2n} \subset \mathbb{R}^D$. They define a preferred frame discussed in section 2.2. We can interpret

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x) \tag{2.15}$$

as Poisson structure on $\mathcal{M}^{2n}$, noting that the Jacobi identity is trivially satisfied. This is indeed the Poisson structure on $\mathcal{M}^{2n}$ whose quantization is given by the matrices X^μ, $\mu = 1,\ldots,2n$, interpreted as quantization of the coordinate functions x^μ on $\mathcal{M}^{2n}$. Conversely, any $\theta^{\mu\nu}(x)$ can be (locally) quantized as (2.15), and provides together with arbitrary "embedding" functions $\phi^i(x)$ a quantization of $\mathcal{M}^{2n} \subset \mathbb{R}^D$ as described above. In particular, the rank of $\theta^{\mu\nu}$ coincides with the dimension of $\mathcal{M}^{2n}$. We denote its inverse matrix with

$$\theta^{-1}_{\mu\nu}(x)\,, \tag{2.16}$$

which defines a symplectic form on $\mathcal{M}^{2n}$. Then the trace is given semi-classically by the volume of the symplectic form,

$$\begin{aligned}(2\pi)^n\, Tr f \;&\sim\; \int d^{2n}x\,\rho(x)\,f \\ \rho(x) \;&=\; (\det\theta^{-1}_{\mu\nu})^{1/2}.\end{aligned} \tag{2.17}$$

We can now extract the semi-classical limit of the matrix model and its physical interpretation. To understand the effective geometry on $\mathcal{M}^{2n}$, consider again a (test-) particle on

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$\mathcal{M}^{2n}$, modeled by some additional scalar field φ (this could be e.g. $su(k)$ components of ϕ^i). The kinetic term must have the form

$$S[\varphi] \equiv -Tr[X^a,\varphi][X^b,\varphi]\eta_{ab} = -Tr\left([X^\mu,\varphi][X^\nu,\varphi]\eta_{\mu\nu} + [\phi^i,\varphi][\phi^j,\varphi]\delta_{ij}\right) \quad (2.18)$$

which in the semi-classical limit can be written as

$$\begin{aligned} S[\varphi] &\sim \frac{1}{(2\pi)^n}\int d^{2n}x\, \rho(x)\, G^{\mu\nu}(x)\partial_\mu\varphi\partial_\nu\varphi \\ &= \frac{1}{(2\pi)^n}\int d^{2n}x\, |\tilde{G}_{\mu\nu}|^{1/2}\, \tilde{G}^{\mu\nu}(x)\partial_\mu\varphi\partial_\nu\varphi\,. \end{aligned} \quad (2.19)$$

which has the correct covariant form, where [2]

$$G^{\mu\nu}(x) = \theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)g_{\mu'\nu'}(x) \quad (2.20)$$

$$\tilde{G}^{\mu\nu}(x) = e^{-\sigma}\, G^{\mu\nu}(x), \quad (2.21)$$

$$\rho\, G^{\mu\nu} = |\tilde{G}_{\mu\nu}|^{1/2}\, \tilde{G}^{\mu\nu}(x). \quad (2.22)$$

Here $g_{\mu\nu}(x)$ is the metric (2.12) induced on $\mathcal{M}^{2n} \subset \mathbb{R}^D$ via pull-back of η_{ab}. This amounts to

$$\rho = |\tilde{G}_{\mu\nu}|^{1/2}\, e^{-\sigma}, \quad (2.23)$$

$$e^{-(n-1)\sigma} = |G_{\mu\nu}|^{1/4}|g_{\mu\nu}(x)|^{-\frac{1}{4}} = \rho\, |g_{\mu\nu}(x)|^{-\frac{1}{2}} \quad (2.24)$$

where we exclude the case $n = 1$ for simplicity. Therefore the kinetic term on $\mathcal{M}^{2n}_\theta$ is governed by the metric $\tilde{G}_{\mu\nu}(x)$, which depends on the Poisson tensor $\theta^{\mu\nu}$ and the embedding ("closed string") metric $g_{\mu\nu}(x)$. Similarly, the matrix model action (2.8) can be written in the semi-classical limit as

$$S_{YM} = -Tr[X^a,X^b][X^{a'},X^{b'}]\eta_{aa'}\eta_{bb'} \sim \frac{4}{(2\pi)^n}\int d^{2n}x\, \rho(x)\eta(x), \quad (2.25)$$

where

$$\eta(y) = \frac{1}{4}G^{\mu\nu}(x)g_{\mu\nu}(x) \sim -\frac{1}{4}[X^a,X^b][X^{a'},X^{b'}]\eta_{aa'}\eta_{bb'}. \quad (2.26)$$

We note that

$$|\tilde{G}_{\mu\nu}(x)| = |g_{\mu\nu}(x)|, \qquad 2n = 4 \quad (2.27)$$

on 4-dimensional branes, hence $\tilde{G}_{\mu\nu}$ is unimodular for $D = 4$ in the preferred matrix coordinates [1]. This means that the Poisson tensor $\theta^{\mu\nu}$ does not enter the Riemannian volume at all. This leads to a very interesting mechanism for "stabilizing flat space", which may hold the key for the cosmological constant problem as discussed in [2].

Some properties. The metric $\tilde{G}^{\mu\nu}$ satisfies the following useful identities:

$$0 = \partial_\mu(\rho\,\theta^{\mu\nu}) = \partial_\mu\left(e^{-\sigma}\sqrt{|\tilde{G}|}\theta^{\mu\nu'}\right) = \sqrt{|\tilde{G}|}\,\tilde{\nabla}_\mu(e^{-\sigma}\theta^{\mu\nu'}) \quad (2.28)$$

$$\tilde{\Gamma}^\mu = -\frac{1}{\rho}e^{-\sigma}\,\partial_\nu(G^{\nu\mu}\,\rho) = -e^{-\sigma}\,\theta^{\nu\nu'}\partial_\nu(\theta^{\mu\eta}g_{\eta\nu'}(x)). \quad (2.29)$$

The first is a consequence of the Jacobi identity resp. (2.23); for the short proof see [2]. (2.29) follows from (2.22) and (2.28).

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Equation of motion for test particle φ. The covariant e.o.m. for φ obtained from the semi-classical action (2.19) is

$$\Delta_{\tilde{G}}\varphi = (\tilde{G}^{\mu\nu}\partial_\mu\partial_\nu - \tilde{\Gamma}^\mu\partial_\mu)\varphi = 0\,. \tag{2.30}$$

These covariant equations of motion can also be derived from the matrix model e.o.m.

$$[X^a,[X^b,\varphi]]\eta_{ab} = 0 \tag{2.31}$$

which follow from (2.18). To see this, we can cast this expression in a covariant form as follows:

$$\begin{aligned}
[X^a,[X^b,\varphi]]\eta_{ab} &= [X^\mu,[X^\nu,\varphi]]\eta_{\mu\nu} + [\phi^i,[\phi^j,\varphi]]\,\delta_{ij} \\
&= i[X^\mu,\theta^{\nu\eta}\partial_\eta\varphi]\eta_{\mu\nu} + i[\phi^i,\theta^{\nu\eta}\partial_\nu\phi^j\partial_\eta\varphi]\,\delta_{ij} \\
&\sim -\theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}\partial_\eta\varphi)\eta_{\mu\nu} - \theta^{\mu\rho}\partial_\mu\phi^i\partial_\rho(\theta^{\nu\eta}\partial_\nu\phi^j\partial_\eta\varphi)\,\delta_{ij} \\
&= -\theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}\partial_\eta\varphi)\eta_{\mu\nu} - \theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}\delta g_{\mu\nu}\partial_\eta\varphi) \\
&= -\theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}g_{\mu\nu}(x))\partial_\eta\varphi - G^{\rho\eta}\partial_\rho\partial_\eta\varphi \\
&= e^\sigma(\tilde{\Gamma}^\eta\partial_\eta\varphi - \tilde{G}^{\rho\eta}\partial_\rho\partial_\eta\varphi) = -e^\sigma\Delta_{\tilde{G}}\varphi
\end{aligned} \tag{2.32}$$

in agreement with (2.30), using (2.28), (2.29) and denoting

$$\delta g_{\mu\nu} \equiv \partial_\mu\phi^i\partial_\nu\phi^j\,\delta_{ij}. \tag{2.33}$$

We will see below that in the preferred coordinates x^μ defined by the matrix model, on-shell geometries satisfy

$$\tilde{\Gamma}^\mu \overset{\text{e.o.m.}}{=} 0, \tag{2.34}$$

and the equations of motion for φ simplify as

$$\tilde{G}^{\rho\eta}\partial_\rho\partial_\eta\varphi = 0. \tag{2.35}$$

However, this non-covariant form holds only for on-shell geometries in the preferred x^μ coordinates.

Equation of motion for X^a. The same argument gives the equations of motion for the embedding functions ϕ^i in the matrix model (2.8),

$$\Delta_{\tilde{G}}\phi^i = 0 \tag{2.36}$$

and similarly for $x^\mu \sim X^\mu$,

$$\Delta_{\tilde{G}}x^\mu = 0. \tag{2.37}$$

This is consistent with the ambiguity of the splitting $X^a = (X^\mu,\phi^i)$ into coordinates and scalar fields. In particular, on-shell geometries (2.38) imply harmonic coordinates, which in General Relativity would be interpreted as gauge condition.

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Equation of motion for $\theta^{-1}_{\mu\nu}(x)$. Reconsider the e.o.m. for the tangential components X^μ from the matrix model (2.8):

$$\begin{aligned} 0 = [X^b,[X^\nu,X^{b'}]]\eta_{bb'} &= [X^\mu,[X^\nu,X^{\mu'}]]\eta_{\mu\mu'} + [\phi^i,[X^\nu,\phi^j]]\,\delta_{ij} \\ &= -\theta^{\mu\rho}\partial_\rho\theta^{\nu\mu'}\eta_{\mu\mu'} - \theta^{\mu\rho}\partial_\mu\phi^i\partial_\rho(\theta^{\nu\eta}\partial_\eta\phi^j\,\delta_{ij}) \\ &= -\theta^{\mu\rho}\partial_\rho(\theta^{\nu\eta}g_{\mu\eta}(x)) \\ &= -e^\sigma\,\tilde\Gamma^\nu \\ &= -\theta^{\nu\nu'}G^{\rho\eta'}(x)\partial_\rho\theta^{-1}_{\nu'\eta'} - \theta^{\nu\eta}\theta^{\mu\rho}\partial_\rho g_{\mu\eta} \end{aligned} \tag{2.38}$$

since $\partial_\rho\delta g_{\mu\eta}(x) = \partial_\rho g_{\mu\eta}(x)$, i.e.

$$G^{\rho\eta}(x)\partial_\rho\theta^{-1}_{\eta\nu} = \theta^{\mu\rho}\partial_\rho g_{\mu\nu}(x). \tag{2.39}$$

As shown in [2], (2.39) together with (2.36) can be written in covariant form as

$$\tilde G^{\gamma\eta}(x)\,\tilde\nabla_\gamma(e^\sigma\theta^{-1}_{\eta\nu}) \;=\; e^{-\sigma}\,\tilde G_{\mu\nu}\,\theta^{\mu\gamma}\,\partial_\gamma\eta(x) \tag{2.40}$$

Here $\tilde\nabla$ denotes the Levi-Civita connection with respect to the effective metric $\tilde G^{\mu\nu}$ (2.21). Remarkably, this equation is also a consequence of a "matrix" Noether theorem due to a basic symmetry (4.1) of the matrix model, as shown in section 4. This means that (2.40) is protected from quantum corrections, and should be taken serious at the quantum level. It provides the relation between the noncommutativity $\theta^{\mu\nu}(x)$ and the metric $\tilde G^{\mu\nu}$.

Eq. (2.40) has the structure of covariant Maxwell equations coupled to an external current,

$$\tilde G^{\gamma\eta}(x)\,\tilde\nabla_\gamma\theta^{-1}_{\eta\nu} \;=\; -\tilde G^{\gamma\eta}(x)\,\theta^{-1}_{\eta\nu}\partial_\gamma\sigma + e^{-2\sigma}\,\tilde G_{\mu\nu}\theta^{\mu\gamma}\,\partial_\gamma\eta(x) \tag{2.41}$$

Therefore the noncommutativity $\theta^{\mu\nu}(x)$ should be completely determined (for given boundary conditions) by the scalar functions $\eta(x)$ and $\rho(x)$. Their physical meaning remains to be elucidated.

2.2 Frame

Define

$$\mathcal{J}^\eta_\gamma = e^{-\sigma/2}\,\theta^{\eta\gamma'}g_{\gamma'\gamma} = -e^{-\sigma/2}\,G^{\eta\gamma'}\theta^{-1}_{\gamma'\gamma}. \tag{2.42}$$

Then the effective metric can be written as

$$\tilde G^{\mu\nu} = \mathcal{J}^\mu_\rho\,\mathcal{J}^\nu_{\rho'}\,g^{\rho\rho'} = -(\mathcal{J}^2)^\mu_\rho\,g^{\rho\nu}, \tag{2.43}$$

which is the reason for choosing the above normalization. Moreover, using (2.24) we have

$$\det\mathcal{J} = 1, \qquad 2n = 4 \tag{2.44}$$

in the case of 4 dimensions, which we assume in this section from now on. Therefore $\mathcal{J}$ should be regarded as a preferred vielbein. However, there is no distinction between

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"Lorentz" and "coordinate" indices here, and neither local Lorentz nor general coordinate transformations are allowed a priori. $\mathcal{J}_\nu^\rho$ satisfies the following properties

$$\begin{aligned} G_{\mu\rho}\mathcal{J}_\nu^\rho &= -e^{-\sigma/2}\,\theta^{-1}_{\mu\nu} \\ (\mathcal{J}^2)_\rho^\mu &= -\tilde{G}^{\mu\nu} g_{\nu\rho} \\ tr\mathcal{J}^2 &= -4e^{-\sigma}\eta \\ \mathcal{J}_\mu^\rho\,\tilde{G}_{\rho\nu} + \mathcal{J}_\nu^\rho\,\tilde{G}_{\mu\rho} &= 0 \end{aligned} \tag{2.45}$$

due to the anti-symmetry of $\theta^{-1}_{\mu\nu}$. Note that $\mathcal{J}_\nu^\rho \in so(\tilde{G})$ due to (2.45).

To make this more explicit, consider (at a point x) a local coordinate system where $g_{\mu\nu} = \mathrm{diag}(1,1,1,1)$ (in the Euclidean case) and $\theta^{\mu\nu}$ has the form

$$\sqrt{\rho}\,\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & -\alpha^{-1} & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}\;; \tag{2.46}$$

this can always be achieved. Then locally

$$\begin{aligned} (\mathcal{J}^2)_\rho^\mu &= -\mathrm{diag}(\alpha^2, \alpha^{-2}, \alpha^{-2}, \alpha^2), \\ \tilde{G}^{\mu\nu} &= \mathrm{diag}(\alpha^2, \alpha^{-2}, \alpha^{-2}, \alpha^2), \end{aligned} \tag{2.47}$$

and

$$e^{-\sigma}\eta = \frac{1}{2}(\alpha^2 + \alpha^{-2}). \tag{2.48}$$

As a consequence, $\mathcal{J}$ satisfies a characteristic equation

$$\begin{aligned} (\mathcal{J}^2 + \alpha^2)(\mathcal{J}^2 + \alpha^{-2}) &= 0, \\ \mathcal{J}^2 + 2e^{-\sigma}\eta + \mathcal{J}^{-2} &= 0 \end{aligned} \tag{2.49}$$

using (2.48). In certain cases $\mathcal{J}$ may play the role of an almost-complex structure, which will be explored elsewhere.

Remarks on Minkowski case and Wick rotation. If $\eta_{\mu\nu}$ has Minkowski signature, there are some rather strange modifications to the above formulae. There are other reasons to believe that the appropriate implementation of Minkowski signature might require more than simply setting $\eta_{\mu\nu} = (-1,1,1,1)$. In addition, one could impose $(X^0)^\dagger = -X^0$, replacing $X^0 \equiv iT$. Then θ^{0i} would be imaginary rather than real. There are several reasons why such a scenario might be preferable, but this remains to be explored.

2.3 SO(D) invariance and normal embedding coordinates

The matrix model action is invariant under SO(D) (resp. SO$(1,D-1)$) rotations as well as translations $X^a \to X^a + c^a\mathbb{1}$, which together form the inhomogeneous Euclidean group ISO(D) (resp. the Poincaré group). Taking advantage of this symmetry, one can choose

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for any given point $p \in \mathcal{M}$ adapted coordinates where the brane is tangential to the plane spanned by the first $2n$ components, i.e. $\partial_\mu \phi^i|_p = 0$. Then the embedding metric satisfies

$$\delta g_{\mu\nu}|_p = \partial_\sigma \delta g_{\mu\nu}|_p = 0. \tag{2.50}$$

We denote such coordinates as "normal embedding coordinates"[4] from now on; they exist for any given point $p \in \mathcal{M}$. Note that these are preferred matrix coordinates $x^a \sim X^a$.

These special coordinates often simplify the analysis of general branes, and allow to reduce many considerations to the case of trivially embedded branes. For example, it gives an easy way to see that the matrix model action in the U(1) case can be written as in (2.25) in terms of

$$4\eta(x) = \{x^a, x^b\}\{x^{a'}, x^{b'}\}\eta_{aa'}\eta_{bb'} = G^{\mu\nu}(x) g_{\mu\nu}(x). \tag{2.51}$$

This is so because $\eta(x)$ can be viewed as a $SO(D)$ scalar, which in normal embedding coordinates reduces to $G^{\mu\nu}(x)\eta_{\mu\nu}$. The unique covariant generalization is as above.

For nonabelian gauge fields, a similar argument strongly suggests that their effective action can be evaluated in normal embedding coordinates, where it should reduce to the action computed in [1] without extra dimensions. However, we will give an independent derivation in section 5 based on the equations of motion.

3. Moyal-Weyl plane and deformations

A particular solution of the e.o.m. (2.9) is given by the 4D Moyal-Weyl quantum plane. Its generators $\bar{X}^\mu$ satisfy

$$[\bar{X}^\mu, \bar{X}^\nu] = i\bar{\theta}^{\mu\nu}\mathbb{1}, \tag{3.1}$$

where $\bar{\theta}^{\mu\nu}$ is a constant antisymmetric tensor. The effective geometry for the Moyal-Weyl plane is indeed flat, given by

$$\begin{aligned} \bar{G}^{\mu\nu} &= \bar{\rho}\,\bar{\theta}^{\mu\mu'}\,\bar{\theta}^{\nu\nu'}\bar{g}_{\mu'\nu'}, \\ \bar{\rho} &= |\bar{\theta}^{-1}_{\mu\nu}|^{1/2} \equiv \Lambda_{\rm NC}^4. \end{aligned} \tag{3.2}$$

where the embedding metric is simply

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu}. \tag{3.3}$$

In the Minkowski case, we can choose coordinates where

$$\bar{G}_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \tag{3.4}$$

and

$$\sqrt{\rho}\,\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -s\alpha \\ 0 & 0 & -\alpha^{-1} & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ s\alpha & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

[4]They coincide with Riemannian normal coordinates w.r.t. the Levi-Civita connection corresponding to $g_{\mu\nu}$, but not in general for $\tilde{\nabla}$ corresponding to the effective metric $\tilde{G}_{\mu\nu}$

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where we admit $s \in \{1, i\}$. If we choose $s = 1, \alpha = 1$, then the original flat background metric is

$$\bar{g}_{\mu\nu} = \rho^{-1}\bar{\theta}^{-1}_{\mu\mu'}\bar{\theta}^{-1}_{\nu\nu'}\bar{G}^{\mu'\nu'} = \mathrm{diag}(1,1,1,-1)\,. \tag{3.6}$$

On the other hand if we choose $s = i$, then $\theta^{0\mu}$ is imaginary and the original flat background metric is

$$\bar{g}_{\mu\nu} = \rho^{-1}\bar{\theta}^{-1}_{\mu\mu'}\bar{\theta}^{-1}_{\nu\nu'}\bar{G}^{\mu'\nu'} = \mathrm{diag}(-1,1,1,1) = \bar{G}_{\mu\nu}\,. \tag{3.7}$$

This illustrates the discussion in section 2.2.

3.1 Linearized metric fluctuations

The Riemann and Ricci tensor for a general effective metric $\tilde{G}_{\mu\nu}$ are defined as usual by

$$\begin{aligned}
\Gamma^{\eta}_{\mu\nu} &= \frac{1}{2}\tilde{G}^{\gamma\eta}\left(\partial_\mu\tilde{G}_{\nu\gamma} + \partial_\nu\tilde{G}_{\gamma\mu} - \partial_\gamma\tilde{G}_{\mu\nu}\right), && (3.8)\\
R^{\delta}_{\mu\nu\gamma} &= \partial_\nu\Gamma^{\delta}_{\mu\gamma} - \partial_\gamma\Gamma^{\delta}_{\mu\nu} + \Gamma^{\rho}_{\mu\gamma}\Gamma^{\delta}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\delta}_{\rho\gamma}, && (3.9)\\
R_{\mu\nu} &= R^{\delta}_{\mu\delta\nu}, && (3.10)\\
R &= \tilde{G}^{\mu\nu}R_{\mu\nu}\,. && (3.11)
\end{aligned}$$

Now consider fluctuations of the geometry around a flat Moyal-Weyl plane $\bar{G}_{\mu\nu}$,

$$\tilde{G}_{\mu\nu} = \bar{G}_{\mu\nu} + h_{\mu\nu}, \qquad h = \bar{G}^{\mu\nu}h_{\mu\nu}. \tag{3.12}$$

Then the linearized Riemann and Ricci tensor are given by

$$\begin{aligned}
R^{(1)\delta}_{\mu\nu\gamma} &= \frac{1}{2}\bar{G}^{\delta\rho}\partial_\nu\partial_\mu h_{\rho\gamma} - \frac{1}{2}\bar{G}^{\delta\rho}\partial_\nu\partial_\rho h_{\mu\gamma} - \frac{1}{2}\bar{G}^{\delta\rho}\partial_\gamma\partial_\mu h_{\rho\nu} + \frac{1}{2}\bar{G}^{\delta\rho}\bar{\partial}_\gamma\partial_\rho h_{\mu\nu}\\
R^{(1)}_{\mu\gamma} &= \partial^\rho\partial_{(\mu}h_{\gamma)\rho} - \frac{1}{2}\partial_\mu\partial_\gamma h - \frac{1}{2}\partial^\delta\partial_\delta h_{\mu\gamma}. && (3.13)
\end{aligned}$$

Now we impose the equations of motion in the preferred (matrix) coordinates, given by (2.38)

$$\begin{aligned}
0 &= \tilde{\Gamma}^\mu \sim \partial_\nu\left(\sqrt{|\tilde{G}_{\eta\sigma}|}\,\tilde{G}^{\nu\mu}\right) && (3.14)\\
0 &= \Delta_{\tilde{G}}\,\phi^i. && (3.15)
\end{aligned}$$

For the fluctuations this amounts to

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0, \tag{3.16}$$

and the expression for the linearized Ricci tensor simplifies as

$$R^{(1)}_{\mu\gamma} = -\frac{1}{2}\partial^\delta\partial_\delta h_{\mu\gamma}. \tag{3.17}$$

In General Relativity, (3.16) would simply amount to a gauge fixing without dynamical content. This is not the case here: The preferred matrix coordinates satisfy constraints, and (3.14) resp. (3.16) is part of their equations of motion.

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We now introduce an explicit parametrization of the fluctuations of the flat Moyal-Weyl plane. Tangential fluctuations of the "covariant coordinates" can be parametrized as

$$X^\mu = \bar{X}^\mu - \bar{\theta}^{\mu\nu} A_\nu(x)\,, \tag{3.18}$$

and the A_ν can be interpreted as U(1) gauge fields on $\mathbb{R}^4_\theta$ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + i[A_\mu, A_\nu]$. The variation of the Poisson tensor is easily seen to be $\delta\theta^{-1}_{\mu\nu} = F_{\mu\nu}$. On the other hand, the variation of the embedding metric (2.12)

$$\delta g_{\mu\nu} = \partial_\mu \delta\phi^i \partial_\nu \delta\phi^j \,\delta_{ij} \tag{3.19}$$

is necessarily 2nd order in $\delta\phi$. Using $\delta g^{\mu\nu} = -\bar{g}^{\mu\mu'} \delta g_{\mu'\nu'} \bar{g}^{\nu\nu'}$, the combined metric fluctuation can be written as

$$\begin{aligned} h_{\mu\nu} &= e^{\bar{\sigma}/2} \left((\bar{\mathcal{J}}^{-1})_\nu^{\mu'} F_{\mu'\mu} + (\bar{\mathcal{J}}^{-1})_\mu^{\nu'} F_{\nu'\nu} - \frac{1}{2}((\bar{\mathcal{J}}^{-1})_\nu^{\rho} F_{\rho\sigma} \bar{G}^{\nu\sigma}) \bar{G}_{\mu\nu} \right) \\ &\quad -(\bar{\mathcal{J}}^{-1})_\mu^{\mu'} (\bar{\mathcal{J}}^{-1})_\nu^{\nu'} \delta g_{\mu'\nu'} + \frac{1}{2}(\bar{g}^{\rho\sigma} \delta g_{\rho\sigma}) \bar{G}_{\mu\nu} \end{aligned} \tag{3.20}$$

since $e^\sigma = \sqrt{|g_{\mu\nu}|/|\theta^{-1}_{\mu\nu}|}$ in 4 dimensions (2.24). As a check, we note that $h = \bar{g}^{\rho\sigma}\delta g_{\rho\sigma}$ depends only on the embedding part, cf. (2.27).

3.2 U(1) fluctuations and gravitational waves

For $\delta g_{\mu\nu} = 0$, the on-shell U(1) fluctuations around flat Moyal-Weyl space satisfy the Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0 \tag{3.21}$$

which implies as usual

$$\partial^\rho \partial_\rho F_{\mu\nu} = 0. \tag{3.22}$$

Together with (3.17) and (3.20), it follows that these U(1) metric fluctuations are Ricci-flat,

$$R_{\mu\nu} = 0 \tag{3.23}$$

which was first observed by Rivelles [5]. This can also be seen as a consequence of $\Delta_{\tilde{G}} X^\mu = 0$. We verify below that the corresponding Riemann tensor is indeed non-trivial, thus obtaining a parametrization of the 2 on-shell degrees of freedom of gravitational waves through U(1) gauge fields. Note also that (3.23) is consistent with the Einstein-Hilbert action induced upon quantization [1].

Linearized Riemann tensor. Consider U(1) fluctuations given by plane waves

$$A_\mu(x) = A_\mu e^{ikx} \tag{3.24}$$

such that $F_{\mu\nu} = i(k_\mu A_\nu - k_\nu A_\mu) \neq 0$. Then

$$\begin{aligned} h_{\mu\nu} &= i(\tilde{k}_\nu A_\mu - k_\mu \tilde{A}_\nu) + i(\tilde{k}_\mu A_\nu - k_\nu \tilde{A}_\mu) - \frac{1}{2} i \bar{G}_{\mu\nu} \,((\tilde{k}_\rho A_\sigma - k_\rho \tilde{A}_\sigma)\bar{G}^{\rho\sigma}) \\ &= +i(\tilde{k}_\mu A_\nu + \tilde{k}_\nu A_\mu) - i(k_\mu \tilde{A}_\nu + k_\nu \tilde{A}_\mu) + i e^{\bar{\sigma}/2}\, \bar{G}_{\mu\nu}\, (\bar{\theta}^{\rho\sigma} k_\rho A_\sigma) \end{aligned} \tag{3.25}$$

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where

$$\tilde{k}_\mu = e^{\bar{\sigma}/2}\,(\mathcal{J}^{-1})_\mu^\rho k_\rho, \tag{3.26}$$

noting that

$$\tilde{k}_\rho A_\sigma \bar{G}^{\rho\sigma} = -e^{\bar{\sigma}/2}\,\bar{\theta}^{\rho\sigma} k_\rho A_\sigma = k_\rho \tilde{A}_\sigma \bar{G}^{\rho\sigma}. \tag{3.27}$$

Observe that the term $i(k_\mu \tilde{A}_\nu + k_\nu \tilde{A}_\mu)$ has the form of a diffeomorphism generated by $\tilde{A}_\nu$, and therefore does not contribute to the Riemann tensor. In particular, the deformation is conformally flat if $\tilde{k}_\mu \sim k_\mu$, which means that k_μ is an eigenvector of $\mathcal{J}$. Using (3.25) for the metric fluctuations gives

$$\begin{aligned} R^{(1)}_{\sigma\mu\nu\gamma} &= \bar{G}_{\delta\sigma}\, R^{(1)\delta}_{\mu\nu\gamma} = \frac{1}{2}\,(\partial_\nu\partial_\mu h_{\sigma\gamma} - \partial_\nu\partial_\sigma h_{\mu\gamma} - (\gamma \leftrightarrow \nu)) \\ &= -\frac{i}{2}\Big((k_\mu \tilde{k}_\sigma - \tilde{k}_\mu k_\sigma)(k_\nu A_\gamma - A_\nu k_\gamma) + (k_\mu A_\sigma - k_\sigma A_\mu)(k_\nu \tilde{k}_\gamma - \tilde{k}_\nu k_\gamma)\Big) \\ &\quad -\frac{i}{2} e^{\bar{\sigma}/2}\,(k_\nu k_\mu \bar{G}_{\sigma\gamma} - k_\nu k_\sigma \bar{G}_{\mu\gamma} - k_\gamma k_\mu \bar{G}_{\sigma\nu} + k_\gamma k_\sigma \bar{G}_{\mu\nu})(\bar{\theta}^{\rho\sigma} k_\rho A_\sigma). \end{aligned} \tag{3.28}$$

This implies in particular

$$R^{(1)}_{\sigma\mu\nu\gamma}\bar{\theta}^{\nu\gamma} = -i(k_\mu A_\sigma - k_\sigma A_\mu)(k_\nu \tilde{k}_\gamma \bar{\theta}^{\nu\gamma}) \;=\; F_{\sigma\mu}\,(k_\nu \tilde{k}_\gamma \bar{\theta}^{\nu\gamma}), \tag{3.29}$$

which vanishes[5] only if $k_\nu \tilde{k}_\gamma \bar{\theta}^{\nu\gamma} = 0$. In that case, let $A_\mu = \tilde{k}_\mu$ and (3.28) gives

$$R^{(1)}_{\sigma\mu\nu\gamma} = -i(k_\mu \tilde{k}_\sigma - \tilde{k}_\mu k_\sigma)(k_\nu \tilde{k}_\gamma - \tilde{k}_\nu k_\gamma) \neq 0 \qquad \text{provided} \quad k \not\sim \tilde{k}. \tag{3.30}$$

Now it is easy to see that $\mathcal{J}$ has no real eigenvector in the Euclidean case (since $\mathcal{J}$ consists essentially of 2 rescaled complex structures as discussed in section 2.2), hence

$$R^{(1)}_{\sigma\mu\nu\gamma} \neq 0. \tag{3.31}$$

The same is true in the Minkowski case, provided we use imaginary $\theta^{0\mu}$ as discussed in section 2.2; this seems to be the better choice. Thus the U(1) metric fluctuations have 2 non-trivial physical degrees of freedom, and therefore describe the physical gravitons on flat $\mathbb{R}^4$. If we insist on real $\theta^{0\mu}$ in the Minkowski case, then there would be a preferred direction $k \sim \tilde{k}$ where only one graviton polarization survives.

3.3 Fluctuations of the embedding

Now consider on-shell fluctuations of the embedding metric. The linearized Ricci tensor for $F_{\mu\nu} = 0$ is

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\Delta_{\bar{G}}\left((\bar{\mathcal{J}}^{-1})_\mu^{\mu'}(\bar{\mathcal{J}}^{-1})_\nu^{\nu'}\delta g_{\mu'\nu'} - \frac{1}{2}(\bar{g}^{\rho\sigma}\delta g_{\rho\sigma})\bar{G}_{\mu\nu}\right) \tag{3.32}$$

using (3.17), which vanishes for

$$\Delta_{\bar{G}}\delta g_{\mu\nu} = 0. \tag{3.33}$$

[5] note that $A \not\sim k$ for nontrivial $h_{\mu\nu}$

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This is a rather restrictive condition, which however does have nontrivial solutions, in particular the plane wave solution

$$\delta\phi^i = p^i\, e^{ikx}, \tag{3.34}$$

so that $\partial_\mu \phi^i = k_\mu\, \phi^i$. Imposing the e.o.m. (2.30) leads to light-like momentum $k_\mu k_\nu \bar{G}^{\mu\nu} = 0$, and $R^{(1)}_{\mu\nu} = 0$ follows. While such plane-wave deformations correspond to trivial (flat) deformations of the embedding metric $g_{\mu\nu}$, they do lead to non-trivial deformations of the effective metric $\tilde{G}_{\mu\nu}$, as verified below by computing the corresponding Riemann tensor.

This result holds for every component ϕ^i, and it may appear at first that there are too many "gravitons". However, recall that the embedding fluctuations only contribute at second-order in $\delta\phi$. Therefore these plane waves cannot simply be superimposed, and should presumably *not* be interpreted as gravitational waves. This suggests that these Ricci-flat embedding fluctuations should be interpreted as gravitational field due to some nontrivial mass distributions; this is expected to happen e.g. for the analog of the Schwarzschild geometry. Moreover, one should keep in mind that only certain combinations of these embedding and U(1) fluctuations may survive at the quantum level; this is suggested by the fact that only the "tangential" e.o.m. are ensured by Noether's theorem (4.9). It would be very important to understand better the relation between the U(1) and the embedding fluctuations.

Linearized Riemann tensor. For $F_{\mu\nu} = 0$, the linearized Riemann tensor is

$$\begin{aligned} R^{(1)}_{\sigma\mu\nu\gamma} &= -\frac{1}{2}\Big((\mathcal{J}^{-1})^{\gamma'}_{\gamma}(\mathcal{J}^{-1})^{\sigma'}_{\sigma}\partial_\nu\partial_\mu\delta g_{\sigma'\gamma'} - (\mathcal{J}^{-1})^{\gamma'}_{\gamma}(\mathcal{J}^{-1})^{\mu'}_{\mu}\partial_\nu\partial_\sigma\delta g_{\mu'\gamma'} - (\gamma\leftrightarrow\nu)\Big) \\ &\quad +\frac{1}{4}\left(\bar{G}_{\sigma\gamma}\partial_\nu\partial_\mu h - \bar{G}_{\sigma\nu}\partial_\gamma\partial_\mu h - \bar{G}_{\mu\gamma}\partial_\nu\partial_\sigma h + \bar{G}_{\mu\nu}\partial_\gamma\partial_\sigma h\right) \end{aligned} \tag{3.35}$$

where $h = \bar{g}^{\mu\nu}\delta g_{\mu\nu}$. For plane wave deformations (3.34), this is

$$\begin{aligned} R^{(1)}_{\sigma\mu\nu\gamma} &= \frac{1}{2}e^{-\sigma}\,(\tilde{k}_\sigma k_\mu - \tilde{k}_\mu k_\sigma)(\tilde{k}_\nu k_\gamma - \tilde{k}_\gamma k_\nu)(p^i p^j \delta_{ij}) \\ &\quad -\frac{1}{4}\Big(\bar{G}_{\sigma\gamma}k_\nu k_\mu - \bar{G}_{\sigma\nu}k_\gamma k_\mu - \bar{G}_{\mu\gamma}k_\nu k_\sigma + \bar{G}_{\mu\nu}k_\gamma k_\sigma\Big)h \end{aligned} \tag{3.36}$$

which is indeed nontrivial for $k \neq \tilde{k}$.

4. Noether theorem and semi-classical conservation laws

The basic matrix model (1.1) enjoys the translational symmetry

$$X^a \to X^a + c^a \mathbb{1} \tag{4.1}$$

As shown in [41, 2], this symmetry leads to the conservation law

$$[X^a, T^{a'c}]\eta_{aa'} = 0 \tag{4.2}$$

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which was also discussed in [42] in a somewhat different context, cf. [43]. It can also be verified directly using the matrix e.o.m. (2.9). Here

$$T^{ab} = [X^a, X^c][X^b, X^{c'}]\eta_{cc'} + [X^b, X^c][X^a, X^{c'}]\eta_{cc'} - \frac{1}{2}\eta^{ab}[X^d, X^c][X^{d'}, X^{c'}]\eta_{dd'}\eta_{cc'} \quad (4.3)$$

is the matrix - "energy-momentum tensor". However, its interpretation and consequences are quite surprising in the present context, as we will show. Note first that the indices run from 1 to D. In the semi-classical or geometrical limit, T^{ab} can be written in terms of the geometrical quantities,

$$\begin{aligned}
T^{\mu\nu} &= [X^\mu, X^\rho][X^\nu, X^{\rho'}]\eta_{\rho\rho'} + [X^\nu, X^\rho][X^\mu, X^{\rho'}]\eta_{\rho\rho'} - \frac{1}{2}\eta^{\mu\nu}[X^\rho, X^\sigma][X^{\rho'}, X^{\sigma'}]\eta_{\rho\rho'}\eta_{\sigma\sigma'} \\
&\quad + [X^\mu, \phi_r][X^\nu, \phi_r] + [X^\nu, \phi_r][X^\mu, \phi_r] - \eta^{\mu\nu}[X^\rho, \phi_r][X^{\rho'}, \phi_r]\eta_{\rho\rho'} - \frac{1}{2}\eta^{\mu\nu}[\phi_r, \phi_s][\phi_r, \phi_s] \\
&\sim -2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y) \\
T^{\mu r} &= [X^\mu, X^\rho][\phi_r, X^{\rho'}]\eta_{\rho\rho'} + [\phi_r, X^\rho][X^\mu, X^{\rho'}]\eta_{\rho\rho'} + [X^\mu, \phi_t][\phi_r, \phi_{t'}]\eta_{tt'} + [\phi_r, \phi_{t'}][X^\mu, \phi_t]\eta_{tt'} \\
&\sim -2\theta^{\rho\rho'}\theta^{\mu\mu'}g_{\rho'\mu'}(y)\partial_\rho\phi_r = -2G^{\rho\nu}\partial_\rho\phi_r \\
T^{rs} &= [\phi_r, X^\rho][\phi_s, X^{\rho'}]\eta_{\rho\rho'} + [\phi_s, X^\rho][\phi_r, X^{\rho'}]\eta_{\rho\rho'} + [\phi_r, \phi_t][\phi_s, \phi_{t'}]\eta_{tt'} + [\phi_s, \phi_{t'}][\phi_r, \phi_{t'}]\eta_{tt'} \\
&\quad - \frac{1}{2}g^{rs}[X^\rho, X^\sigma][X^{\rho'}, X^{\sigma'}]\eta_{\rho\rho'}\eta_{\sigma\sigma'} \\
&\sim -2G^{\mu\nu}\partial_\mu\phi_r\partial_\nu\phi_s + 2\eta^{rs}\eta(y)
\end{aligned} \quad (4.4)$$

where

$$\begin{aligned}
g_{\mu\nu}(y) &= \eta_{\mu\nu} + \partial_\mu\phi_r\partial_\nu\phi_r = \eta_{\mu\nu} + \delta g_{\mu\nu}, \\
G^{\mu\nu}(y) &= \theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu\nu}(y) \\
\eta(y) &= \frac{1}{4}\,G^{\mu\nu}g_{\mu\nu}(y) = \frac{1}{4}\,g_{\rho\sigma}(y)\theta^{\rho\rho'}\theta^{\sigma\sigma'}g_{\rho'\sigma'}(y)
\end{aligned} \quad (4.5)$$

as introduced before. We want to elaborate the semi-classical meaning of the conservation laws

$$[X^{\mu'}, T^{\mu\nu}]\eta_{\mu'\mu} + [\phi_r, T^{r\nu}] = 0 \quad (4.6)$$

$$[X^{\mu'}, T^{\mu s}]\bar{\eta}_{\mu'\mu} + [\phi_r, T^{rs}] = 0, \quad (4.7)$$

which we will refer to as tangential resp. scalar conservation law. Using

$$\begin{aligned}
i\frac{1}{2}[\phi_r, T^{r\nu}] &= \theta^{\mu\eta}\partial_\mu\phi_r\partial_\eta(G^{\nu\rho}\partial_\rho\phi_r) = \theta^{\mu\eta}\partial_\eta(G^{\nu\rho}\delta g_{\mu\rho}), \\
-i[X^\nu, T^{\mu s}]\bar{\eta}_{\nu\mu} &= \theta^{\nu\sigma}\partial_\sigma(-2\theta^{\rho\rho'}\theta^{\mu\mu'}\eta_{\rho'\mu'}\partial_\rho\phi_s)\eta_{\nu\mu}, \\
-i[\phi_r, T^{rs}] &= -2\theta^{\eta\sigma}\partial_\sigma(G^{\mu\nu}\delta g_{\eta\mu}\partial_\nu\phi_s) + 2\theta^{\eta\sigma}\partial_\eta\phi_s\partial_\sigma\eta(y)
\end{aligned} \quad (4.8)$$

the tangential conservation law (4.6) gives

$$\begin{aligned}
0 &= -i[X^{\mu'}, T^{\mu\nu}]\bar{g}_{\mu'\mu} - i[\phi_r, T^{r\nu}] \\
&= \theta^{\mu'\rho}\partial_\rho(-2G^{\mu\nu} + 2\bar{g}^{\mu\nu}\eta(y))\bar{g}_{\mu'\mu} - 2\theta^{\mu\eta}\partial_\eta(G^{\nu\rho}\delta g_{\mu\rho}) \\
&= 2\theta^{\nu\rho}\partial_\rho\eta(y) - 2\theta^{\mu\eta}\partial_\eta(G^{\nu\rho}g_{\mu\rho}(y))
\end{aligned} \quad (4.9)$$

which using $\partial_\eta \theta^{\eta\mu} = \theta^{\mu\eta}\rho^{-1}\partial_\eta\rho$ (2.28) can be written as

$$\partial_\eta(\rho\hat{\theta}^{\eta\nu}) = \rho\,\theta^{\rho\nu}\partial_\rho\eta(y) \tag{4.10}$$

introducing the antisymmetric (!) matrix

$$\hat{\theta}^{\nu\eta} = G^{\nu\rho}g_{\rho\mu}(y)\theta^{\mu\eta} = -G^{\nu\rho}\theta^{-1}_{\rho\mu}G^{\mu\eta} = -\theta^{\nu\mu'}g_{\mu'\eta'}\theta^{\eta'\eta}g_{\eta\nu'}\theta^{\nu'\eta} = -e^{\sigma}(\mathcal{J}^2)^{\nu}{}_{\mu}\theta^{\mu\eta} \tag{4.11}$$

which we will encounter again. Similarly, the scalar conservation law (4.7) gives

$$\begin{aligned}
-i[X^\nu, T^{\mu s}]\eta_{\nu\mu} - i[\phi_r, T^{rs}] &= -2\theta^{\eta\sigma}\partial_\sigma(G^{\rho\mu}\partial_\rho\phi_s)\eta_{\eta\mu} \\
&\quad -2\theta^{\eta\sigma}\partial_\sigma(G^{\rho\mu}\delta g_{\eta\mu}\partial_\rho\phi_s) + 2\theta^{\eta\sigma}\partial_\eta\phi_s\partial_\sigma\eta(y) \\
&= -2\theta^{\eta\sigma}G^{\mu\rho}g_{\eta\mu}(y)\partial_\sigma\partial_\rho\phi_s \\
&\quad -2\theta^{\eta\sigma}\partial_\sigma(G^{\mu\rho}g_{\eta\mu}(y))\partial_\rho\phi_s + 2\theta^{\rho\sigma}\partial_\sigma\eta(y)\partial_\rho\phi_s \\
&= -2\theta^{\eta\sigma}g_{\eta\mu}(y)G^{\mu\rho}\partial_\sigma\partial_\rho\phi_s = -2\hat{\theta}^{\rho\sigma}\partial_\sigma\partial_\rho\phi_s = 0
\end{aligned}$$

using the tangential conservation law (4.9) in the last step, and noting that $\hat{\theta}^{\mu\nu}$ is antisymmetric. Hence the scalar conservation law is a consequence of the tangential one. This is somewhat surprising, but reflects the $2n$-dimensional nature of the sub-manifold $\mathcal{M} \subset \mathbb{R}^D$.

The tangential conservation law (4.10) can be rewritten using $|\tilde{G}_{\mu\nu}|^{1/2} = \rho e^\sigma$ (2.23) and $\partial_\eta(\rho\hat{\theta}^{\eta\nu}) = \sqrt{|\tilde{G}|}\,\tilde{\nabla}_\mu(e^{-\sigma}\hat{\theta}^{\mu\nu})$ as follows:

$$\tilde{\nabla}_\mu(e^{-\sigma}\hat{\theta}^{\mu\nu}) = e^{-\sigma}\theta^{\mu\nu}\partial_\mu\eta(y). \tag{4.12}$$

Using (4.11), this coincides precisely with the e.o.m. (2.40)

$$\tilde{\nabla}^\mu(e^\sigma\theta^{-1}_{\mu\nu}) = e^{-\sigma}\tilde{G}_{\nu\nu'}\theta^{\nu'\mu}\partial_\mu\eta(y).$$

It is easy to verify the integrability condition for this conservation law

$$\tilde{\nabla}_\nu\tilde{\nabla}_\mu(e^{-\sigma}\hat{\theta}^{\mu\nu}) = \tilde{\nabla}_\nu(e^{-\sigma}\theta^{\mu\nu}\partial_\mu\eta(y)). \tag{4.13}$$

Since the above derivation is based on an exact conservation law of the underlying matrix model, (4.12) can be trusted also at the quantum level. Indeed the integration measure for the matrix model quite trivially respects the translational symmetry, hence no "anomalies" are expected.

Some results for the embedding metric $g_{\mu\nu}$. The above conservation law (4.12) implies also some interesting results for the embedding metric, notably

$$\tilde{\nabla}^\rho g_{\rho\mu} = 2\partial_\mu(e^{-\sigma}\eta) \tag{4.14}$$

$$\theta^{\mu\rho}\tilde{\nabla}_\rho(e^\sigma g_{\mu\eta}) = \tilde{G}_{\nu\nu'}\theta^{\nu'\mu}\partial_\mu\eta(y) \tag{4.15}$$

$$(\Delta_{\tilde{G}}x^a)\,\partial_\nu x^b\eta_{ab} = 0. \tag{4.16}$$

The last relation becomes more transparent in normal embedding coordinates, where it reduces to $\Delta_{\tilde{G}}x^\mu = 0$. This means that $\Delta_{\tilde{G}}\vec{x}$ is normal to $T\mathcal{M}$ using the metric η_{ab}. This is somewhat weaker than (but a consequence of) the "bare" matrix model equations $\Delta_{\tilde{G}}\phi^i = \tilde{\Gamma}^\mu = 0$, which would imply harmonic embeddings for all coordinates x^a; see also appendix B of [2].

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proof of (4.14) : To see this, consider the following identity:

$$\begin{aligned}\partial_\rho \eta &= \frac{1}{4}\tilde{G}^{\mu\nu}\tilde{\nabla}_\rho(e^\sigma g_{\mu\nu}) \\ &= \frac{1}{2}\tilde{\nabla}_\rho\theta^{\mu\nu}(g\theta g)_{\mu\nu} + \frac{1}{2}\tilde{\nabla}_\rho g_{\mu\nu}G^{\mu\nu} \\ &= \frac{1}{2}\tilde{\nabla}_\rho\theta^{\mu\nu}(g\theta g)_{\mu\nu} + 2\partial_\rho\eta - 2\eta e^{-\sigma}\partial_\rho e^\sigma \end{aligned} \tag{4.17}$$

which implies

$$(g\theta g)_{\alpha\beta}\tilde{\nabla}_\mu\theta^{\alpha\beta} = -2\partial_\mu\eta + 4\eta e^{-\sigma}\partial_\mu e^\sigma = -2e^{2\sigma}\partial_\mu(\eta e^{-2\sigma}) \tag{4.18}$$

$$2e^{2\sigma}\theta^{\nu\mu}\partial_\mu(\eta e^{-2\sigma}) = -(g\theta g)_{\alpha\beta}\theta^{\nu\mu}\tilde{\nabla}_\mu\theta^{\alpha\beta} = -2(gG)^\mu_\alpha\tilde{\nabla}_\mu\theta^{\nu\alpha} \tag{4.19}$$

$$\begin{aligned}\theta^{\nu\mu}\partial_\mu\eta &= (gG)^\mu_\alpha\tilde{\nabla}_\mu\theta^{\alpha\nu} + \eta\theta^{\nu\mu}e^{-2\sigma}\partial_\mu e^{2\sigma} \\ &= e^\sigma\tilde{\nabla}_\mu(e^{-\sigma}\hat{\theta}^{\mu\nu}) + \theta^{\nu\alpha}G^{\mu\rho}\tilde{\nabla}_\mu g_{\rho\alpha} + 2\eta\theta^{\nu\mu}e^{-\sigma}\partial_\mu e^\sigma\end{aligned} \tag{4.20}$$

using the Jacobi identity in the 2nd line. This identity together with the on-shell conservation law (4.12) implies (4.14).

proof of (4.16) : this follows simply from (2.40) together with

$$e^{-\sigma}\theta^{\mu\rho}\tilde{\nabla}_\rho(e^\sigma g_{\mu\eta}) = -\tilde{\nabla}_\rho(\theta^{\rho\mu}g_{\mu\eta}) = \tilde{\nabla}_\rho(G^{\rho\mu}\theta^{-1}_{\mu\eta}) = \tilde{G}^{\rho\mu}\tilde{\nabla}_\rho(e^\sigma\theta^{-1}_{\mu\eta}). \tag{4.21}$$

proof of (4.16) : consider

$$\begin{aligned}\partial^\mu g_{\mu\nu} &= \tilde{G}^{\mu\rho}\partial_\mu\partial_\rho x^a\partial_\nu x^b\eta_{ab} + \tilde{G}^{\mu\rho}\partial_\mu x^a\partial_\rho\partial_\nu x^b\eta_{ab} \\ &\sim (\Delta_{\tilde{G}}x^a + \tilde{\Gamma}^\rho\partial_\rho x^a)\partial_\nu x^b\eta_{ab} + \frac{1}{2}\tilde{G}^{\mu\rho}\partial_\nu g_{\mu\rho} \\ &= \Delta_{\tilde{G}}x^a\,\partial_\nu x^b\eta_{ab} + \tilde{\Gamma}^\rho g_{\rho\nu} + 2\partial_\nu(e^{-\sigma}\eta) - \frac{1}{2}(\partial_\nu\tilde{G}^{\mu\rho})g_{\mu\rho}\end{aligned} \tag{4.22}$$

(where x^a are treated as scalar fields for $a, b = 1, \ldots, D$). Now for any symmetric tensor $g_{\mu\nu}$, one has

$$\begin{aligned}\tilde{\nabla}^\mu g_{\mu\nu} &= \partial^\mu g_{\mu\nu} - \tilde{\Gamma}^\sigma g_{\sigma\nu} + \tilde{G}^{\mu\rho}\tilde{\Gamma}^\sigma_{\rho\nu}g_{\mu\sigma} \\ &= \partial^\mu g_{\mu\nu} - \tilde{\Gamma}^\sigma g_{\sigma\nu} + \frac{1}{2}g_{\mu\rho}\partial_\nu\tilde{G}^{\mu\rho}\end{aligned} \tag{4.23}$$

hence

$$\tilde{\nabla}^\mu g_{\mu\nu} = \Delta_{\tilde{G}}x^a\,\partial_\nu x^b\eta_{ab} + 2\partial_\nu(e^{-\sigma}\eta) \tag{4.24}$$

which together with (4.14) gives (4.16).

5. Nonabelian gauge fields

To minimize notational conflicts, we denote the basic dynamical matrices with Y^a in this section. Consider the same matrix model as above

$$S_{\rm YM} = -Tr[Y^a, Y^b][Y^{a'}, Y^{b'}]\eta_{aa'}\eta_{bb'} \tag{5.1}$$

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but for matrix backgrounds of the form

$$Y^a = \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{cases} X^\mu \otimes \mathbb{1}_n, & a = \mu = 1, 2, \dots, 2n, \\ \phi^i \otimes \mathbb{1}_n, & a = 2n + i, \ i = 1, \dots, D - 2n. \end{cases} \tag{5.2}$$

We want to understand general fluctuations around the above background. Since the U(1) components describe the geometry, we expect to find $su(n)$-valued gauge fields as well as scalar fields in the adjoint. It turns out that the following gives an appropriate parametrization of these general fluctuations:

$$\begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{pmatrix} X^\mu \otimes \mathbb{1}_n + \mathcal{A}^\mu \\ \phi^i \otimes \mathbb{1}_n + \Phi^i + \mathcal{A}^\rho \partial_\rho (\phi^i \otimes \mathbb{1}_n + \Phi^i) \end{pmatrix} \sim (1 + \mathcal{A}^\nu \partial_\nu) \begin{pmatrix} X^\mu \otimes \mathbb{1}_n \\ \phi^i \otimes \mathbb{1}_n + \Phi^i \end{pmatrix} \tag{5.3}$$

where

$$\begin{aligned} \mathcal{A}^\mu &= \mathcal{A}^\mu_\alpha \otimes \lambda^\alpha \ = \ -\theta^{\mu\nu} A_{\nu,\alpha} \otimes \lambda^\alpha, \\ \Phi^i &= \Phi^i_\alpha \otimes \lambda^\alpha \end{aligned} \tag{5.4}$$

parametrize the $su(n)$-valued gauge fields resp. scalar fields, and λ^α denotes the generators of $su(n)$. This amounts to the leading term in a Seiberg-Witten (SW) map [11], which relates noncommutative and commutative gauge theories with the appropriate gauge transformations[6]. This SW parametrization can be characterized by requiring that the noncommutative gauge transformation $\delta_{nc} Y^a = i[Y^a, \Lambda]$ induces for the $su(n)$ components $A_\mu = A_{\mu,\alpha}\lambda^\alpha$ and Φ^i the ordinary gauge transformations

$$\begin{aligned} \delta_{cl} A_\mu &= i[A_\mu, \Lambda]_{su(n)} + \partial_\mu \Lambda(x) \quad + O(\theta) \\ \delta_{cl} \Phi^i &= i[\Phi^i, \Lambda]_{su(n)} \quad + O(\theta^2), \end{aligned} \tag{5.5}$$

cf. [44]. Here the subscript $[\Phi^i, \Lambda]_{su(n)}$ indicates that only the commutator of the explicit $su(n)$ generators is to be taken, but not the $O(\theta)$ contributions from the Poisson bracket. This strongly suggests that the matrix model action expressed in terms of these A_μ should reduce to a conventional gauge theory in the semi-classical limit; this was verified in [1].

In our context, the SW map (5.3) can be understood geometrically in a very simple way. Recall that the U(1) sector (i.e. the components proportional to $\mathbb{1}_n$) describes the geometrical degrees of freedom: in the geometrical limit, $X^a = (X^\mu, \phi^i)$ become functions

$$x^a = (x^\mu, \phi^i(x)) \tag{5.6}$$

on $\mathcal{M}$ which describe the embedding of the $2n$-dimensional brane $\mathcal{M} \subset \mathbb{R}^D$. Then the one-form $A_\mu dx^\mu$ together with the Poisson tensor determines a tangential vector field

$$A_\mu e^\mu = A_\mu \theta^{\mu\nu} \partial_\nu = \mathcal{A}^\nu \partial_\nu \quad \in T_p\mathcal{M}, \tag{5.7}$$

[6] We could drop the term $\mathcal{A}^\rho \partial_\rho \Phi^i$ for the nonabelian scalars at leading order as long as they do not acquire a VEV (in which case they would contribute to the geometry), however the term $\mathcal{A}^\rho \partial_\rho \phi^i$ does contribute to the effective action for A_μ.

whose push-forward in the ambient space $\mathbb{R}^D$

$$\mathcal{A}^\nu \partial_\nu x^a \cong \mathcal{A}^\nu(\delta^\mu_\nu, \partial_\nu \phi^i) \tag{5.8}$$

coincides with the fluctuations $\delta X^a = Y^a - X^a$ of the dynamical matrices in (5.3) (for vanishing Φ^i). This provides the link between gauge fields and "covariant coordinates" X^a. The nonabelian scalar fields Φ^i can be thought of as coordinate functions of $\mathcal{M}$ embedded in further extra dimensions corresponding to $su(n)$. They behave as scalar fields, but might contribute to the background geometry if they acquire a non-trivial VEV, completely analogous to the U(1) components ϕ^i.

After this preparation, we claim that the effective action for $su(n)$-valued gauge fields A_μ on general $\mathcal{M}^{2n}_\theta \subset \mathbb{R}^D$ in the matrix model (5.1) in the semi-classical limit is

$$\begin{aligned} S_{\rm YM}[\mathcal{A}] &= Tr\Big(G^{\mu\mu'}G^{\nu\nu'}F_{\mu\nu}\,F_{\mu'\nu'} - F_{\mu'\nu'}F_{\mu\nu}\hat\theta^{\mu'\nu'}\theta^{\mu\nu} - 2F_{\mu'\mu}F_{\nu'\nu}\hat\theta^{\mu'\nu'}\theta^{\nu\mu} \\ &\qquad + \frac{1}{2}\eta\theta^{\mu\nu}\theta^{\mu'\nu'}(F_{\mu\nu}F_{\mu'\nu'} + 2F_{\mu\nu'}F_{\nu\mu'})\Big) \\ &\sim \int d^{2n}x\,|\tilde G_{\mu\nu}|^{1/2}e^{\sigma}\,\tilde G^{\mu\mu'}\tilde G^{\nu\nu'}\mathrm{tr}(F_{\mu\nu}\,F_{\mu'\nu'}) \;-\; S_{\rm NC} \end{aligned} \tag{5.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the $su(n)$-valued field strength, and

$$\begin{aligned} S_{\rm NC} &= S_{\rm NC,1} + S_{\rm NC,2} \\ &= \int d^{2n}x\,\rho\,\mathrm{tr}\Big((F\theta)(F\hat\theta) - 2(F\theta F\hat\theta)\Big) - \frac{1}{2}\int d^{2n}x\,\eta\rho\,\mathrm{tr}\Big((F\theta)(F\theta) - 2(F\theta F\theta)\Big) \\ &\overset{n=2}{=} -2\int \eta(x)\,\mathrm{tr}F\wedge F\,. \end{aligned} \tag{5.10}$$

Note that the embedding metric g enters the "would-be topological term" $S_{\rm NC}$ only through η resp. $\hat\theta^{\mu\nu}$. The contraction of the tensor indices is spelled out in (5.9). The last line holds for 4-dimensional branes, and can be seen using

$$\frac{1}{2}(F\wedge F)_{\mu\nu\rho\sigma}\hat\theta^{\mu\nu}\theta^{\rho\sigma} = (F_{\mu\nu}\hat\theta^{\mu\nu})(F_{\rho\sigma}\theta^{\rho\sigma}) + 2F_{\mu\sigma}F_{\nu\rho}\hat\theta^{\mu\nu}\theta^{\rho\sigma} \tag{5.11}$$

and $\hat\theta\wedge\theta = \eta(y)\theta\wedge\theta$, see [1]. For the 4-dimensional matrix model this was first obtained through a direct but rather non-transparent computation of the action [1], requiring the 2nd order Seiberg Witten map. We will show below that (5.9) holds also on general non-trivially embedded $2n$ -dimensional NC branes $\mathcal{M}\subset\mathbb{R}^D$, by showing that the corresponding equations of motion coincide with the equation (5.32) derived from the covariant conservation law (4.6), which in turn follow from the basic matrix equations of motion (2.9).

Nonabelian scalars. It follows easily along the lines of section 2.1 that the geometrical action for the $su(n)$ -valued scalars Φ^i_α in the matrix model at leading order is

$$S_{\rm YM}[\Phi] \sim 2\int d^4x\,\sqrt{|\tilde G|}\,\tilde G^{\mu\nu}\mathrm{tr}(D_\mu\Phi^i D_\nu\Phi^i + \frac{1}{2}e^{-\sigma}\,[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]\delta_{ii'}\delta_{jj'}), \tag{5.12}$$

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where $D_\mu \Phi^i = \partial_\mu \Phi^i + i[A_\mu, \Phi^i]$. We will indeed verify that the corresponding equations of motion follow from (2.9). It is worth pointing out that in the case $D = 10$ and upon adding suitable fermions as in the IKKT model [12], this will lead to the analog of $N = 4$ SYM theory, albeit on a general background geometry. The SUSY transformations then act non-trivially on the geometry, cf. the discussion in [4].

Equations of motion. We now derive the equations of motion for the gauge field A_μ from the effective action (5.9). Consider first

$$\begin{aligned}\delta S_{\mathrm{NC},2} &= -\int \rho\eta((\delta F\theta)(F\theta) - 2tr(\delta F\theta F\theta) \\ &= 2\int \delta A_\nu D_\mu\Big(\rho\eta F_{\sigma\eta}(\theta^{\mu\nu}\theta^{\sigma\eta} - 2\theta^{\nu\sigma}\theta^{\eta\mu})\Big) \\ &= 2\int \rho\delta A_\nu F_{\sigma\eta}\,\partial_\mu\eta(\theta^{\mu\nu}\theta^{\sigma\eta} - 2\theta^{\eta\mu}\theta^{\nu\sigma}) \end{aligned} \tag{5.13}$$

where $D_\mu = \partial_\mu + i[A_\mu, .]$, using the identity $\partial_\mu(\rho\theta^{\mu\nu}) = 0$ (2.28), the Bianci identities for F:

$$(\theta^{\mu\nu}\theta^{\sigma\eta} - 2\theta^{\nu\sigma}\theta^{\eta\mu})D_\mu F_{\sigma\eta} = \theta^{\mu\nu}\theta^{\sigma\eta}\Big(-D_\sigma F_{\eta\mu} - D_\eta F_{\mu\sigma}\Big) - 2\theta^{\nu\sigma}\theta^{\eta\mu}D_\mu F_{\sigma\eta} = 0, \tag{5.14}$$

and the Jacobi identity for $\theta^{\mu\nu}$ which implies

$$\begin{aligned}&\int \rho\eta\delta A_\nu F_{\sigma\eta}\Big(\theta^{\mu\nu}\partial_\mu\theta^{\sigma\eta} - 2\theta^{\eta\mu}\partial_\mu\theta^{\nu\sigma}\Big) \\ &\qquad = \int \rho\eta\delta A_\nu F_{\sigma\eta}\Big(-\theta^{\mu\sigma}\partial_\mu\theta^{\eta\nu} - \theta^{\mu\eta}\partial_\mu\theta^{\nu\sigma} + 2\theta^{\mu\eta}\partial_\mu\theta^{\nu\sigma}\Big) = 0. \end{aligned} \tag{5.15}$$

Now consider the terms $\theta \wedge \hat\theta$

$$\begin{aligned}\delta S_{\mathrm{NC},1} &= \int \rho(\delta F\theta)(F\hat\theta) - 2\mathrm{tr}(\delta F\theta F\hat\theta) + (\delta F\hat\theta)(F\theta) - 2tr(\delta F\hat\theta F\theta) \\ &= 2\int \delta A_\nu D_\mu\Big(\rho F_{\sigma\eta}(\theta^{\nu\mu}\hat\theta^{\sigma\eta} + \hat\theta^{\nu\mu}\theta^{\sigma\eta} + 2\theta^{\nu\sigma}\hat\theta^{\eta\mu} + 2\hat\theta^{\nu\sigma}\theta^{\eta\mu})\Big) \\ &= 2\int \delta A_\nu \tilde\nabla^A_\mu\Big(\rho F_{\sigma\eta}(\theta^{\nu\mu}\hat\theta^{\sigma\eta} + \hat\theta^{\nu\mu}\theta^{\sigma\eta} + 2\theta^{\nu\sigma}\hat\theta^{\eta\mu} + 2\hat\theta^{\nu\sigma}\theta^{\eta\mu})\Big)\end{aligned}$$

where $\tilde\nabla^A_\mu = \tilde\nabla_\mu + i[A_\mu, .]$, using the fact that expression in brackets is completely antisymmetric in $\mu\sigma\eta$. Putting this together gives the e.o.m.

$$\begin{aligned}0 &= 2\sqrt{|\tilde G|}\,\tilde\nabla^A_\rho\,(e^\sigma F^{\rho\nu}) + \rho F_{\sigma\eta}\,\partial_\mu\eta(\theta^{\mu\nu}\theta^{\sigma\eta} - 2\theta^{\eta\mu}\theta^{\nu\sigma}) \\ &\quad + \tilde\nabla^A_\mu\Big(\rho F_{\sigma\eta}(\theta^{\nu\mu}\hat\theta^{\sigma\eta} + \hat\theta^{\nu\mu}\theta^{\sigma\eta} + 2\theta^{\nu\sigma}\hat\theta^{\eta\mu} + 2\hat\theta^{\nu\sigma}\theta^{\eta\mu})\Big). \end{aligned} \tag{5.16}$$

This matches precisely with (5.32), which is derived from the matrix equations of motion (2.9) resp. the conservation law (4.6).

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5.1 Covariant Yang-Mills equations from matrix model

Consider the matrix equations of motion for

$$Y^\mu = X^\mu + \mathcal{A}^\mu = X^\mu - \theta^{\mu\nu}(x)A_\nu \tag{5.17}$$

where $A_\nu = A_{\nu,\alpha}\lambda^\alpha$ is a $su(n)$ gauge field, setting $\Phi^i = 0$ but keeping ϕ^i. Then

$$\begin{aligned} -i[Y^\mu, Y^\nu] &= \theta^{\mu\nu} + \theta^{\mu\rho}\partial_\rho\mathcal{A}^\nu - \theta^{\nu\rho}\partial_\rho\mathcal{A}^\mu - i[\mathcal{A}^\mu, \mathcal{A}^\nu] \\ &= (1 + \mathcal{A}\cdot\partial)\theta^{\mu\nu} + \mathcal{F}^{\mu\nu} \end{aligned} \tag{5.18}$$

where

$$\mathcal{F}^{\mu\nu} = -\theta^{\mu\mu'}\theta^{\nu\nu'}F_{\mu'\nu'} \tag{5.19}$$

denoting $\mathcal{A}\cdot\partial\phi \equiv \mathcal{A}^\rho\partial_\rho\phi$. Furthermore, we note the following useful formulae

$$[Y, \phi^i + \mathcal{A}^\rho\partial_\rho\phi^i] = [Y, Y^\rho]\partial_\rho\phi^i + i\mathcal{A}^\rho\{Y, \partial_\rho\phi^i\} \tag{5.20}$$

$$\begin{aligned} [Y^\mu, \phi^i + \mathcal{A}^\rho\partial_\rho\phi^i] &= [Y^\mu, Y^\rho]\partial_\rho\phi^i + i\mathcal{A}^\rho\{Y^\mu, \partial_\rho\phi^i\} \\ &= i((1+\mathcal{A}\cdot\partial)\theta^{\mu\rho})\partial_\rho\phi^i - i\theta^{\mu\mu'}\theta^{\rho\rho'}F_{\mu'\rho'}\partial_\rho\phi^i + i\mathcal{A}^\rho\theta^{\mu\eta}\partial_\eta\partial_\rho\phi^i \\ &= i(1+\mathcal{A}\cdot\partial)(\theta^{\mu\rho}\partial_\rho\phi^i) + i\mathcal{F}^{\mu\rho}\partial_\rho\phi^i \end{aligned} \tag{5.21}$$

$$\begin{aligned} [(1+\mathcal{A}^\eta\partial_\eta)\phi^i, (1+\mathcal{A}^\rho\partial_\rho)\phi^j] &= [(1+\mathcal{A}^\eta\partial_\eta)\phi^i, Y^\rho]\partial_\rho\phi^j + i\mathcal{A}^\rho\{(1+\mathcal{A}^\eta\partial_\eta)\phi^i, \partial_\rho\phi^j\} \\ &= -\left(i(1+\mathcal{A}\cdot\partial)(\theta^{\rho\eta}\partial_\eta\phi^i) + i\mathcal{F}^{\rho\eta}\partial_\eta\phi^i\right)\partial_\rho\phi^j + i\mathcal{A}^\rho\{\phi^i, \partial_\rho\phi^j\} \\ &= i(1+\mathcal{A}\cdot\partial)(\theta^{\eta\rho}\partial_\eta\phi^i\partial_\rho\phi^j) + i\mathcal{F}^{\eta\rho}\partial_\eta\phi^i\partial_\rho\phi^j \end{aligned} \tag{5.22}$$

which hold to $O(\theta^2)$. We can now obtain the commutative limit of T^{ab} (4.3) for nonabelian gauge fields:

$$\begin{aligned} T^{\mu\nu} &= [Y^\mu, Y^\rho][Y^\nu, Y^{\rho'}]\eta_{\rho\rho'} + [Y^\nu, Y^\rho][Y^\mu, Y^{\rho'}]]\eta_{\rho\rho'} - \frac{1}{2}g^{\mu\nu}[Y^\rho, Y^\sigma][Y^{\rho'}, Y^{\sigma'}]\eta_{\rho\rho'}\eta_{\sigma\sigma'} \\ &\quad +[Y^\mu, (1+\mathcal{A}\cdot\partial)\phi_r][Y^\nu, (1+\mathcal{A}\cdot\partial)\phi_r] + [Y^\nu, (1+\mathcal{A}\cdot\partial)\phi_r][Y^\mu, (1+\mathcal{A}\cdot\partial)\phi_r] \\ &\quad -\eta^{\mu\nu}[Y^\rho, (1+\mathcal{A}\cdot\partial)\phi_r][Y^{\rho'}, (1+\mathcal{A}\cdot\partial)\phi_r]\eta_{\rho\rho'} \\ &\quad -\frac{1}{2}\eta^{\mu\nu}[(1+\mathcal{A}\cdot\partial)\phi_r, (1+\mathcal{A}\cdot\partial)\phi_s][(1+\mathcal{A}\cdot\partial)\phi_r, (1+\mathcal{A}\cdot\partial)\phi_s] \\ &\sim (1+\mathcal{A}\cdot\partial)(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y)) \\ &\quad -2\mathcal{F}^{\mu\rho}\theta^{\nu\rho'}\eta_{\rho\rho'} - 2\mathcal{F}^{\mu\rho}\theta^{\nu\rho'}\delta g_{\rho\rho'} - 2\theta^{\mu\rho}\mathcal{F}^{\nu\rho'}\eta_{\rho\rho'} - 2\theta^{\mu\rho}\mathcal{F}^{\nu\rho'}\delta g_{\rho\rho'} \\ &\quad +\eta^{\mu\nu}\left(\eta_{\rho'\sigma'}\theta^{\rho\rho'}\mathcal{F}^{\sigma\sigma'}\eta_{\rho\sigma} + 2\theta^{\rho\rho'}\mathcal{F}^{\sigma\sigma'}\eta_{\rho'\sigma'}\delta g_{\rho\sigma} + \delta g_{\rho'\sigma'}\theta^{\rho\rho'}\mathcal{F}^{\sigma\sigma'}\delta g_{\rho\sigma}\right) \\ &= (1+\mathcal{A}\cdot\partial)(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y)) \\ &\quad -2\mathcal{F}^{\mu\rho}\theta^{\nu\rho'}g_{\rho\rho'}(x) - 2\theta^{\mu\rho}\mathcal{F}^{\nu\rho'}g_{\rho\rho'}(x) + \eta^{\mu\nu}(g_{\rho'\sigma'}(x)\theta^{\rho\rho'}\mathcal{F}^{\sigma\sigma'}g_{\rho\sigma}(x)) \\ &= (1+\mathcal{A}\cdot\partial)(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y)) - 2\mathcal{G}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}(\mathcal{G}^{\alpha\beta}g_{\alpha\beta}) \\[1ex] T^{\mu r} &= -[Y^\mu, Y^\rho][Y^{\rho'}, (1+\mathcal{A}\cdot\partial)\phi_r]\eta_{\rho\rho'} - [Y^\rho, (1+\mathcal{A}\cdot\partial)\phi_r][Y^\mu, Y^{\rho'}]\eta_{\rho\rho'} \\ &\quad +[Y^\mu, (1+\mathcal{A}\cdot\partial)\phi_t][(1+\mathcal{A}\cdot\partial)\phi_r, (1+\mathcal{A}\cdot\partial)\phi_{t'}]\delta_{tt'} \\ &\quad +[(1+\mathcal{A}\cdot\partial)\phi_r, (1+\mathcal{A}\cdot\partial)\phi_{t'}][Y^\mu, (1+\mathcal{A}\cdot\partial)\phi_t]\delta_{tt'} \\ &\sim -2(1+\mathcal{A}\cdot\partial)\theta^{\rho\rho'}\theta^{\mu\mu'}g_{\rho'\mu'}(y)\partial_\rho\phi_r - 2(\mathcal{F}^{\rho\rho'}\theta^{\mu\mu'}g_{\rho'\mu'}(x) + \theta^{\rho\rho'}\mathcal{F}^{\mu\mu'}g_{\rho'\mu'}(x))\partial_\rho\phi_r \\ &= -2(1+\mathcal{A}\cdot\partial)(G^{\rho\mu}\partial_\rho\phi_r) - 2\mathcal{G}^{\mu\nu}\partial_\rho\phi_r \end{aligned} \tag{5.23}$$

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where we introduce the "nonabelian metric"

$$\mathcal{G}^{\mu\nu} = \theta^{\mu\alpha} F_{\alpha\beta} G^{\beta\nu} + \theta^{\nu\alpha} F_{\alpha\beta} G^{\mu\beta} = \mathcal{G}^{\nu\mu} \tag{5.24}$$

and note that

$$\hat{\theta}^{\mu\nu} F_{\mu\nu} = -\frac{1}{2} \mathcal{G}^{\mu\nu} g_{\mu\nu}. \tag{5.25}$$

This gives

$$\begin{aligned} -i[Y^{\mu'}, T^{\mu\nu}]\eta_{\mu'\mu} &= -i[Y^{\mu'}, (1 + \mathcal{A} \cdot \partial)(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y))]\eta_{\mu'\mu} \\ &\quad -\theta^{\mu'\rho}(\partial_\rho + i[A_\rho, .])\Big(2\mathcal{G}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}(\mathcal{G}^{\alpha\beta} g_{\alpha\beta})\Big)\eta_{\mu'\mu} \\ &= (1 + \mathcal{A} \cdot \partial)\left(\theta^{\mu'\rho}\partial_\rho(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y))\right)\eta_{\mu'\mu} \\ &\quad +\mathcal{F}^{\mu'\rho}\partial_\rho(-2G^{\mu\nu} + 2\eta^{\mu\nu}\eta(y))\eta_{\mu'\mu} \\ &\quad -\theta^{\mu'\rho}(\partial_\rho + i[A_\rho, .])\Big(2\mathcal{G}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}(\mathcal{G}^{\alpha\beta} g_{\alpha\beta})\Big)\eta_{\mu'\mu} \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} i\frac{1}{2}[(1+\mathcal{A}\cdot\partial)\phi_r, T^{r\nu}] &= (1 + \mathcal{A} \cdot \partial)\left(\theta^{\eta\rho}\partial_\eta\phi_r\partial_\rho(G^{\mu\nu}\partial_\mu\phi_r)\right) + \mathcal{F}^{\eta\rho}\partial_\eta\phi_r\partial_\rho(G^{\mu\nu}\partial_\mu\phi_r) \\ &\quad -i[(1 + \mathcal{A} \cdot \partial)\phi_r, \mathcal{G}^{\nu\mu}\partial_\mu\phi_r] \\ &= (1+\mathcal{A}\cdot\partial)\left(\theta^{\eta\rho}\partial_\rho(G^{\mu\nu}\delta g_{\eta\mu})\right) + \mathcal{F}^{\eta\rho}\partial_\rho(G^{\mu\nu}\delta g_{\eta\mu}) - i[Y^\sigma, \mathcal{G}^{\nu\rho}\partial_\rho\phi_r]\partial_\sigma\phi_r \\ &= (1+\mathcal{A}\cdot\partial)\left(\theta^{\eta\rho}\partial_\rho(G^{\mu\nu}\delta g_{\eta\mu})\right) + \mathcal{F}^{\eta\rho}\partial_\rho(G^{\mu\nu}\delta g_{\eta\mu}) + \theta^{\sigma\eta} D_\eta\left(\mathcal{G}^{\nu\mu}\delta g_{\mu\sigma}\right) \end{aligned}$$

to $O(\theta^2)$, where $D_\mu = \partial_\mu + i[A_\mu, .]$, using (5.20). Hence the "tangential" conservation law (4.6) gives

$$\begin{aligned} 0 &= -i[Y^{\mu'}, T^{\mu\nu}]\eta_{\mu'\mu} - i[Y^r, T^{r\nu}] \\ &= 2(1 + \mathcal{A} \cdot \partial)\left(\theta^{\nu\rho}\partial_\rho\eta(y) - \theta^{\eta\rho}\partial_\rho(G^{\rho\nu} g_{\eta\rho})\right) - 2\mathcal{F}^{\eta\rho}\partial_\rho(G^{\mu\nu} g_{\eta\mu}) + 2\mathcal{F}^{\nu\rho}\partial_\rho\eta(y) \\ &\quad -\theta^{\eta\rho} D_\rho\Big(2\mathcal{G}^{\mu\nu} g_{\eta\mu} - \frac{1}{2}\delta^\nu_\eta(\mathcal{G}^{\alpha\beta} g_{\alpha\beta})\Big). \end{aligned} \tag{5.27}$$

Using the geometrical conservation law (4.9), this reduces to

$$0 = -\mathcal{F}^{\eta\rho}\partial_\rho(G^{\mu\nu} g_{\eta\mu}) + \mathcal{F}^{\nu\rho}\partial_\rho\eta(y) - \theta^{\eta\rho} D_\rho\Big(\mathcal{G}^{\mu\nu} g_{\eta\mu} - \frac{1}{4}\delta^\nu_\eta(\mathcal{G}^{\alpha\beta} g_{\alpha\beta})\Big) \tag{5.28}$$

Now we note that

$$\begin{aligned} \theta^{\eta\rho} D_\rho(\mathcal{G}^{\mu\nu} g_{\eta\mu}) &= -\rho^{-1} D_\rho(\rho\theta^{\rho\eta} g_{\eta\mu}\mathcal{G}^{\mu\nu}) \\ &= -\rho^{-1} D_\rho(-\rho G^{\rho\alpha} F_{\alpha\beta} G^{\beta\nu} + \rho\hat{\theta}^{\rho\beta} F_{\beta\alpha}\theta^{\alpha\nu}) \\ &= \rho^{-1}\sqrt{|\tilde{G}|}\,(\tilde{\nabla}_\rho + i[A_\rho, .])(e^\sigma F^{\rho\nu}) - \rho^{-1} D_\rho(\rho\hat{\theta}^{\rho\beta} F_{\beta\alpha}\theta^{\alpha\nu}). \end{aligned} \tag{5.29}$$

Similarly,

$$\frac{1}{4}\theta^{\nu\rho} D_\rho(\mathcal{G}^{\alpha\beta} g_{\alpha\beta}) = -\frac{1}{2}\theta^{\nu\rho} D_\rho(\hat{\theta}^{\alpha\beta} F_{\alpha\beta}) = -\frac{1}{2}\rho^{-1} D_\rho(\rho\theta^{\nu\rho}\hat{\theta}^{\alpha\beta} F_{\alpha\beta}) \tag{5.30}$$

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and

$$\begin{aligned}
\rho\mathcal{F}^{\eta\rho}\partial_\rho(G^{\mu\nu}g_{\eta\mu}) &= \theta^{\eta\alpha}F_{\alpha\beta}\partial_\rho(\rho\theta^{\beta\rho}G^{\mu\nu}g_{\eta\mu})\\
&= F_{\alpha\beta}\partial_\rho(\rho\theta^{\beta\rho}G^{\nu\mu}g_{\mu\eta}\theta^{\eta\alpha}) - \rho F_{\alpha\beta}g_{\eta\mu}G^{\mu\nu}\theta^{\beta\rho}\partial_\rho\theta^{\eta\alpha}\\
&= F_{\alpha\beta}\partial_\rho(\rho\theta^{\beta\rho}\hat\theta^{\nu\alpha}) + \frac{1}{2}\rho F_{\alpha\beta}g_{\eta\mu}G^{\mu\nu}\theta^{\eta\rho}\partial_\rho\theta^{\alpha\beta}\\
&= F_{\alpha\beta}\partial_\rho(\rho\theta^{\beta\rho}\hat\theta^{\nu\alpha}) + \frac{1}{2}\rho F_{\alpha\beta}\hat\theta^{\nu\rho}\partial_\rho\theta^{\alpha\beta}\\
&= F_{\alpha\beta}\partial_\rho\left(\rho\theta^{\beta\rho}\hat\theta^{\nu\alpha} + \frac{1}{2}\rho\hat\theta^{\nu\rho}\theta^{\alpha\beta}\right) - \frac{1}{2}\rho F_{\alpha\beta}\theta^{\nu\rho}\theta^{\alpha\beta}\partial_\rho\eta\\
&= D_\rho\left(\rho F_{\alpha\beta}\left(\theta^{\beta\rho}\hat\theta^{\nu\alpha} + \frac{1}{2}\hat\theta^{\nu\rho}\theta^{\alpha\beta}\right)\right) - \frac{1}{2}\rho F_{\alpha\beta}\theta^{\nu\rho}\theta^{\alpha\beta}\partial_\rho\eta \qquad (5.31)
\end{aligned}$$

since

$$2F_{\alpha\beta}\theta^{\beta\rho}\partial_\rho\theta^{\eta\alpha} = -F_{\alpha\beta}\theta^{\eta\rho}\partial_\rho\theta^{\alpha\beta}$$

using the Jacobi identity, and the Bianci identity

$$D_\rho F_{\alpha\beta}\left(\theta^{\beta\rho}\hat\theta^{\nu\alpha} + \frac{1}{2}\hat\theta^{\nu\rho}\theta^{\alpha\beta}\right) = -\hat\theta^{\nu\alpha}\theta^{\beta\rho}(D_\alpha F_{\beta\rho} + D_\beta F_{\rho\alpha} + D_\rho F_{\alpha\beta}) = 0\,.$$

Therefore (5.28) becomes

$$\begin{aligned}
0 &= -\rho^{-1}\sqrt{|\tilde G|}\,(\tilde\nabla_\rho + i[A_\rho, .])(e^\sigma F^{\rho\nu}) + \frac{1}{2}F_{\alpha\beta}(2\theta^{\nu\alpha}\theta^{\beta\rho} + \theta^{\nu\rho}\theta^{\alpha\beta})\partial_\rho\eta\\
&\quad + \frac{1}{2}\rho^{-1}D_\rho\Big(\rho F_{\alpha\beta}(2\theta^{\nu\alpha}\hat\theta^{\rho\beta} + \theta^{\nu\rho}\hat\theta^{\beta\alpha} + 2\hat\theta^{\nu\alpha}\theta^{\rho\beta} + \hat\theta^{\nu\rho}\theta^{\beta\alpha})\Big)\\
&= -\rho^{-1}\sqrt{|\tilde G|}\,\tilde\nabla^A_\rho(e^\sigma F^{\rho\nu}) + \frac{1}{2}F_{\alpha\beta}(2\theta^{\nu\alpha}\theta^{\beta\rho} + \theta^{\nu\rho}\theta^{\alpha\beta})\partial_\rho\eta\\
&\quad + \frac{1}{2}\rho^{-1}\tilde\nabla^A_\rho\Big(\rho F_{\alpha\beta}(2\theta^{\nu\alpha}\hat\theta^{\rho\beta} + \theta^{\nu\rho}\hat\theta^{\beta\alpha} + 2\hat\theta^{\nu\alpha}\theta^{\rho\beta} + \hat\theta^{\nu\rho}\theta^{\beta\alpha})\Big) \qquad (5.32)
\end{aligned}$$

where $\tilde\nabla^A_\mu = \tilde\nabla_\mu + i[A_\mu, .]$, using the fact that expression in brackets is completely antisymmetric in $\alpha\beta\rho$. This is precisely the e.o.m. (5.16) derived from the effective action (5.9).

It is remarkable that this e.o.m. is derived from an underlying symmetry of the model. This means that it is protected from quantum corrections. In particular, the coefficients of the would-be topological term is fixed and cannot change under renormalization. While this is formally true even in the $D = 4$ model, the quantization probably makes sense only in the $D = 10$ model.

One might object that the equations of motion do not determine or rule out a "truly-topological" term $\sim \int F \wedge F$. However, such a term could come in the $D = 10$ model only through $\int \rho F F\theta\theta$, which has the wrong scaling dimension in θ and is therefore excluded. Hence the action is determined uniquely to be of the form (5.9), (5.10).

This mechanism for determining the would-be topological action is quite remarkable and should have profound consequences once a semi-realistic vacuum with non-trivial low-energy gauge group is found. This can also be seen in connection with the finiteness of the $N = 4$ SYM theory, where the so-called θ-angle is protected from running under renormalization by supersymmetry; this in turn is related to Montonen-Olive duality [45, 46].

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Nonabelian scalar fields. Finally, the action (5.12) implies the following equations of motion for the nonabelian scalars

$$0 = \tilde{G}^{\mu\nu}(\tilde{\nabla}_\nu + i[A_\nu, .])D_\mu\Phi^k + e^{-\sigma}\,[\Phi^i, [\Phi^k, \Phi^j]]\delta_{ij}. \tag{5.33}$$

This e.o.m. is easily derived from the basic matrix e.o.m. (2.9), which is simply a nonabelian version of (2.32): keeping only contributions up to $O(\theta^2)$, we have

$$\begin{aligned}
[Y^a, [Y^b, \Phi^k]]\eta_{ab} &= [Y^\mu, [Y^\nu, \Phi^k]]\eta_{\mu\nu} + [\phi^i \otimes 1\!\!1_n + \Phi^i, [\phi^j \otimes 1\!\!1_n + \Phi^j, \Phi^k]]\,\delta_{ij} \\
&\sim -\theta^{\mu\rho}\,(\partial_\rho + i[A_\rho, .])\left(\theta^{\nu\eta}(\partial_\eta + i[A_\eta, .])\Phi^k\right)\eta_{\mu\nu} \\
&\quad -\{\phi^i, \theta^{\nu\eta}\partial_\nu\phi^j\partial_\eta\Phi^k\}\,\delta_{ij} + [\Phi^i, [\Phi^j, \Phi^k]]\,\delta_{ij} \\
&= -\theta^{\mu\rho}\,(\partial_\rho + i[A_\rho, .])\left(\theta^{\nu\eta}(\partial_\eta + i[A_\eta, .])\Phi^k\right)\eta_{\mu\nu} \\
&\quad -\rho^{-1}\partial_\rho\left(\rho\theta^{\mu\rho}\theta^{\nu\eta}\delta g_{\mu\nu}\partial_\eta\Phi^k\right) + [\Phi^i, [\Phi^j, \Phi^k]]\,\delta_{ij} \\
&= -G^{\rho\eta}(\partial_\rho + i[A_\rho, .])(\partial_\eta + i[A_\eta, .])\Phi^k \\
&\quad -e^\sigma\tilde{\Gamma}^\eta(\partial_\eta + i[A_\eta, .])\Phi^k + [\Phi^i, [\Phi^j, \Phi^k]]\,\delta_{ij} \\
&= -e^\sigma\tilde{G}^{\rho\eta}(\tilde{\nabla}_\rho + i[A_\rho, .])(\partial_\eta + i[A_\eta, .])\Phi^k + [\Phi^i, [\Phi^j, \Phi^k]]\,\delta_{ij}
\end{aligned}$$

using (2.38). This involves the expected gauge-covariant Laplacian. It would be interesting to compute also higher-order corrections in θ, which might involve curvature contributions.

In principle, the e.o.m. for the nonabelian gauge fields (5.32) could also be derived in a similar way. However this is manageable only in normal embedding coordinates (2.50), i.e. taking advantage of the underlying SO(D) symmetry of the model. The derivation presented above based on Noethers theorem is not only more profound but also considerably simpler.

6. Discussion and conclusion

In this paper, the foundations of the matrix-model approach to (emergent) gravity are developed further. The effective action for nonabelian gauge fields on non-trivially embedded branes $\mathcal{M}_\theta \subset \mathbb{R}^D$ is obtained, which turns out to be the expected generalization of the result in [1] for flat embeddings. This shows that the effective metric $\tilde{G}_{\mu\nu}$ (2.21) governs also the nonabelian gauge fields, and must therefore be interpreted as gravitational metric. Since $\tilde{G}_{\mu\nu}$ depends on the embedding metric $g_{\mu\nu}$, we can indeed obtain the most general effective metric in this framework for $D \geq 10$ using standard embedding theorems [15], at least locally. Therefore the $D = 10$ matrix model, notably the IKKT model, is in principle capable of describing generic 4-dimensional metrics. The mechanism of emergent gravity under consideration here becomes a serious candidate for a realistic theory of gravity.

Furthermore, we have shown that the equations of motion for the Poisson tensor $\theta^{\mu\nu}$ and the nonabelian gauge fields are consequences of a Noether theorem, and therefore protected from quantum corrections. This is very remarkable, and supports the viability of this framework at the quantum level. We also elaborated in some more detail the metric fluctuations around the flat Moyal-Weyl background. This leads to a simplified

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understanding of the Ricci-flatness of the U(1) modes as first observed in [5], interpreted as gravitational waves. On the other hand, one of the main open problems is to find an analog of the Schwarzschild solution. Since the Einstein-Hilbert action is induced only upon quantization, this can be addressed reliably only once the one-loop effective action is known. However, the validity of the geometric equation (2.40) at the quantum level established here should be a useful guide already at this point.

Even though the physical properties of emergent gravity are not well understood, there are several striking aspects which make this framework very attractive. First, its definition requires no classical-geometrical notions of geometry whatsoever. The geometry arises dynamically, and in fact physical condensation and melting processes of the geometry have been observed in similar toy models [47]. This is very appealing from the point of view of quantum gravity and cosmology. Another fascinating aspect of the present framework is that it leads to a unified picture of gravitons and nonabelian gauge fields, which arise as abelian resp. nonabelian fluctuations of the basic matrices (covariant coordinates) around a geometrical background. The quantization around a such a background is technically rather straightforward, similar to nonabelian gauge field theory. Remarkably, flat space remains to be a solution at one loop, cf. [1, 2] and a related discussion in [7]. In this context, further analytical and numerical work on the emergence of 4-dimensional NC branes in matrix models is needed, particularly for the supersymmetric case; see also [48, 49] for work in this context.

We conclude by stressing that the gravity "theory" under consideration is not based on some theoretical expectations, but is simply the result of a careful analysis of the semi-classical limit of this type of matrix models. Clearly much more work is required, and it seems likely that many more surprising results and insights are waiting to be discovered.

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Acknowledgments

I would like to thank in particular H. Grosse for useful discussions and support, and P. Aschieri, L. Freidel, D. Klammer, K. Krasnov, J. Madore, M. Buric, P. Schreivogl and P. Schupp for discussions on various related topics. This work was supported by the FWF project P20017.

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